SOLVING REAL ANALYSIS
Solutions to Problems Suggested by Robert G. Bartle

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INTERNET ARCHIVE

This is a work of math problems. Hints, solutions and answers are the products of the author's efforts. Typos and mistakes are totally unavoidable.

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Preface

The present volume contains hints or full solutions to many of the exercises in the first seven chapters of the text *Elements of Real Analysis*, 2nd Edition, [Bar91] by Robert G. Bartle, *Professor of Mathematics*, *University of Illinois Urbana-Champaign*. Solving problems being an essential part of the learning process, my goal is to provide those learning and teaching mathematical analysis, real analysis, or the theory of functions with a large number of worked out exercises. The following theorem may give an indication as to the level of treatment given in Bartle's book: Bolzano-Weierstrass theorem, Heine-Borel theorem, Cantor intersection theorem, Lebesgue covering theorem, Baire's theorem, fixed point theorem for contradictions, Stone-Weierstrass theorem, Tietze extension theorem, Arzelà–Ascoli theorem, Rie representation theorem (for bounded positive linear functionals), Cauchy-Hadamard theorem, and Bernstein's theorem. Bartle's textbook could well provides a sound basis for a course taught at the undergraduate level for second or third year math students who have studied calculus. Therefore this solutions manual can be helpful to anyone learning or teaching mathematical analysis at the undergraduate level.

As the presentation of Bartle's textbook is so condensed that many readers would find it off-putting, I encourage the reader to work through all of the exercises. To make the solutions concise, I have included only the necessary arguments; the reader may have to fill in the details to get complete proofs.

Comments and questions on possibly erroneous solutions, as well as suggestions for more elegant or more complete solutions will be greatly appreciated.

Huy Bui
Georgia Tech, 2019

Contents

Part 1

Introduction: A Glimpse at Set Theory

CHAPTER 1

The Algebra of Sets

EXERCISE (1.A). Establish statement (d) of Theorem 1.5.

SOLUTION. Let x be an element of $A \cup (B \cap C)$, then $x \in A$ or $x \in B \cap C$. This means that $x \in A$, or both $x \in B$ and $x \in C$. Hence we both have (i) $x \in A$ or $x \in B$, and (ii) $x \in A$ or $x \in C$. Therefore, $x \in A \cup B$ and$x \in A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$. This shows that $A \cup (B \cap C)$ is a subset of$(A \cup B) \cap (A \cup C)$. Conversely, let y be an element of $(A \cup B) \cap (A \cup C)$. Then, both (iii) $y \in A \cup B$, and (iv) $y \in A \cup C$. It follows that $y \in A$, or both $y \in B$ and $y \in C$. Therefore, $y \in A$ or $y \in B \cap C$ so that $y \in A \cup (B \cap C)$. Hence $(A \cup B) \cap (A \cup C)$ is a subset of $A \cup (B \cap C)$. In view of Definition 1.1, we conclude that the sets $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$ are equal.

EXERCISE (1.B). Prove that $A \subseteq B$ if and only if $A \cap B = A$.

EXERCISE (1.C). Show that the set D of elements which belong either to A or B but not to both is given by

$$D = (A \backslash B) \cup (B \backslash A)$$

This set D is often called the **symmetric difference** of A and B. Represent it by a diagram.

EXERCISE (1.D). Show that the symmetric difference D of A and B is also given by $D = (A \cup B) \backslash (A \cap B)$.

SOLUTION. Suppose $x \in (A \backslash B) \cup (B \backslash A)$. Then $x \in A \backslash B$ or $x \in B \backslash A$. If $x \in A \backslash B$, then x is in A but x is not in B. Hence x is in A, but x is not $A \cap B$. That is, $x \in A \backslash (A \cap B)$. Similarly, if $x \in B \backslash A$, then $x \in B \backslash (A \cap B)$. Therefore, $x \in A \backslash (A \cap B)$ or $x \in B \backslash (A \cap B)$, showing that $x \in (A \cup B) \backslash (A \cap B)$.

Conversely, suppose $x \in (A \cup B) \backslash (A \cap B)$. Then x is in $A \cup B$ but x is not in $A \cap B$. Thus x is in $A \backslash (A \cap B)$ or x is in $B \backslash (A \cap B)$. It follows that $x \in A \backslash B$ or $x \in B \backslash A$, so that $x \in (A \backslash B) \cup (B \backslash A)$.

Since the sets $(A \backslash B) \cup (B \backslash A)$ and $(A \cup B) \backslash (A \cap B)$ contains the same elements, they are equal by Definition 1.1.

EXERCISE (1.E). If $B \subseteq A$, show that $B = A\backslash(A\backslash B)$.

EXERCISE (1.F.). If A and B are any sets, show that $A \cap B = A\backslash(A\backslash B)$.

EXERCISE (1.I.). If $\{A_1, A_2, \ldots, A_n\}$ is a collection of sets, and if E is any sets, show that

$$E \cap \bigcup_{j=1}^{n} A_j = \bigcup_{j=1}^{n} (E \cap A_j), \qquad E \cup \bigcup_{j=1}^{n} A_j = \bigcup_{j=1}^{n} (E \cup A_j).$$

SOLUTION. Suppose $x \in E \cap \bigcup_{j=1}^{n} A_j$. Then $x \in E$ and $x \in \bigcup_{j=1}^{n} A_j$. Hence $x \in E$ and $x \in A_{j_0}$ for some $j_0 \in \{1, 2, \ldots, n\}$. That is, $x \in E \cap A_{j_0}$. Since $E \cap A_{j_0} \subset \bigcup_{j=1}^{n} (E \cap A_j)$, so $x \in \bigcup_{j=1}^{n} (E \cap A_j)$, showing that $E \cap \bigcup_{j=1}^{n} A_j \subset \bigcup_{j=1}^{n} (E \cap A_j)$. Conversely, suppose $x \in \bigcup_{j=1}^{n} (E \cap A_j)$. Then $x \in E \cap A_{j_0}$ for some $j_0 \in \{1, 2, \ldots, n\}$. That is, $x \in E$ and $x \in A_{j_0}$. Since $A_{j_0} \subset \bigcup_{j=1}^{n} A_j$, so $x \in \bigcup_{j=1}^{n} A_j$. It follows that $x \in E \cap \bigcup_{j=1}^{n} A_j$, so that $\bigcup_{j=1}^{n} (E \cap A_j) \subset E \cap \bigcup_{j=1}^{n} A_j$. Since the sets $E \cap \bigcup_{j=1}^{n} A_j$ and $\bigcup_{j=1}^{n} (E \cap A_j)$ contains the same elements, they are equal by Definition 1.1.

Suppose $x \in E \cup \bigcup_{j=1}^{n} A_j$. Then $x \in E$ or $x \in \bigcup_{j=1}^{n} A_j$. Hence $x \in E$ or $x \in A_{j_0}$ for some $j_0 \in \{1, 2, \ldots, n\}$. That is, $x \in E \cup A_{j_0}$. Since $E \cup A_{j_0} \subset \bigcup_{j=1}^{n} (E \cup A_j)$, so $x \in \bigcup_{j=1}^{n} (E \cup A_j)$, showing that $E \cup \bigcup_{j=1}^{n} A_j \subset \bigcup_{j=1}^{n} (E \cup A_j)$. Conversely, suppose $x \in \bigcup_{j=1}^{n} (E \cup A_j)$. Then $x \in E \cup A_{j_0}$ for some $j_0 \in \{1, 2, \ldots, n\}$. That is, $x \in E$ or $x \in A_{j_0}$. Since $A_{j_0} \subset \bigcup_{j=1}^{n} A_j$, so $x \in \bigcup_{j=1}^{n} A_j$. It follows that $x \in E \cup \bigcup_{j=1}^{n} A_j$, so that $\bigcup_{j=1}^{n} (E \cup A_j) \subset E \cup \bigcup_{j=1}^{n} A_j$. Since the sets $E \cup \bigcup_{j=1}^{n} A_j$ and $\bigcup_{j=1}^{n} (E \cup A_j)$ contains the same elements, they are equal by Definition 1.1.

Exercise (1.J.). If $\{A_1, A_2, \ldots, A_n\}$ is a collection of sets, and if E is any sets, show that

$$E \cap \bigcap_{j=1}^{n} A_j = \bigcap_{j=1}^{n} (E \cap A_j), \qquad E \cup \bigcap_{j=1}^{n} A_j = \bigcap_{j=1}^{n} (E \cup A_j).$$

Solution. Suppose $x \in E \cap \bigcap_{j=1}^{n} A_j$. Then $x \in E$ and $x \in \bigcap_{j=1}^{n} A_j$. Hence $x \in E$ and $x \in A_j$ for all $j \in \{1, 2, \ldots, n\}$. That is, $x \in E \cap A_j$ for all $j \in \{1, 2, \ldots, n\}$. Therefore, $x \in \bigcap_{j=1}^{n} (E \cap A_j)$, showing that $E \cap \bigcap_{j=1}^{n} A_j \subset \bigcap_{j=1}^{n} (E \cap A_j)$. Conversely, suppose $x \in \bigcap_{j=1}^{n} (E \cap A_j)$. Then $x \in E \cap A_{j_0}$ for all $j \in \{1, 2, \ldots, n\}$. That is, $x \in E$ and $x \in A_j$ for all $j \in \{1, 2, \ldots, n\}$. Thus $x \in E$ and $x \in \bigcap_{j=1}^{n} A_j$. It follows that $x \in E \cap \bigcap_{j=1}^{n} A_j$, so that $\bigcap_{j=1}^{n} (E \cap A_j) \subset E \cap \bigcap_{j=1}^{n} A_j$. Since the sets $E \cap \bigcap_{j=1}^{n} A_j$ and $\bigcap_{j=1}^{n} (E \cap A_j)$ contains the same elements, they are equal by Definition 1.1.

Suppose $x \in E \cup \bigcap_{j=1}^{n} A_j$. Then $x \in E$ or $x \in \bigcap_{j=1}^{n} A_j$. Hence $x \in E$ or $x \in A_j$ for all $j \in \{1, 2, \ldots, n\}$. That is, $x \in E \cup A_j$ for all $j \in \{1, 2, \ldots, n\}$. Therefore, $x \in \bigcap_{j=1}^{n} (E \cup A_j)$, showing that $E \cup \bigcap_{j=1}^{n} A_j \subset \bigcap_{j=1}^{n} (E \cup A_j)$. Conversely, suppose $x \in \bigcap_{j=1}^{n} (E \cup A_j)$. Then $x \in E \cup A_{j_0}$ for all $j \in \{1, 2, \ldots, n\}$. That is, $x \in E$ or $x \in A_j$ for all $j \in \{1, 2, \ldots, n\}$. Thus $x \in E$ or $x \in \bigcap_{j=1}^{n} A_j$. It follows that $x \in E \cup \bigcap_{j=1}^{n} A_j$, so that $\bigcap_{j=1}^{n} (E \cup A_j) \subset E \cup \bigcap_{j=1}^{n} A_j$. Since the sets $E \cup \bigcap_{j=1}^{n} A_j$ and $\bigcap_{j=1}^{n} (E \cup A_j)$ contains the same elements, they are equal by Definition 1.1.

Exercise (1.K.). Let E be a set and $\{A_1, A_2, \ldots, A_n\}$ be a collection of sets. Establish the De Morgan laws:

$$E \backslash \bigcap_{j=1}^{n} A_j = \bigcup_{j=1}^{n} (E \backslash A_j), \qquad E \backslash \bigcup_{j=1}^{n} A_j = \bigcap_{j=1}^{n} (E \backslash A_j).$$

Note that if $E\backslash A$; is denoted by $C(A_i)$, these relations take the form

$$\mathscr{C}(\bigcap_{j=1}^{n} A_j) = \bigcup_{j=1}^{n} \mathscr{C}(A_j), \qquad \mathscr{C}(\bigcup_{j=1}^{n} A_j) = \bigcap_{j=1}^{n} \mathscr{C}(A_j).$$

SOLUTION. Suppose $x \in E\backslash \bigcap_{j=1}^{n} A_j$. Then $x \in E$ and $x \notin \bigcap_{j=1}^{n} A_j$. Hence $x \in E$ and $x \notin A_{j_0}$ for some $j_0 \in \{1, 2, \ldots, n\}$. This implies $x \in E\backslash A_{j_0}$ for some $j_0 \in \{1, 2, \ldots, n\}$. Since $E\backslash A_{j_0} \subset \bigcup_{j=1}^{n} (E\backslash A_j)$, so $x \in \bigcup_{j=1}^{n} (E\backslash A_j)$, showing that $E\backslash \bigcap_{j=1}^{n} A_j \subset \bigcup_{j=1}^{n} (E\backslash A_j)$. Conversely, suppose $x \in \bigcup_{j=1}^{n} (E\backslash A_j)$. Then $x \in E\backslash A_{j_0}$ for some $j_0 \in \{1, 2, \ldots, n\}$. Thus $x \in E$ and $x \notin A_{j_0}$ for some $j_0 \in \{1, 2, \ldots, n\}$. This implies $x \in E$ and $x \notin \bigcap_{j=1}^{n} A_j$. It follows that $x \in E\backslash \bigcap_{j=1}^{n} A_j$, so that $\bigcup_{j=1}^{n} (E\backslash A_j) \subset E\backslash \bigcap_{j=1}^{n} A_j$. Since the sets $E\backslash \bigcap_{j=1}^{n} A_j$ and $\bigcup_{j=1}^{n} (E\backslash A_j)$ contains the same elements, they are equal by Definition 1.1.

Suppose $x \in E\backslash \bigcup_{j=1}^{n} A_j$. Then $x \in E$ and $x \notin \bigcup_{j=1}^{n} A_j$. Hence $x \in E$ and $x \notin A_j$ for all $j \in \{1, 2, \ldots, n\}$. This implies $x \in E\backslash A_j$ for all $j \in \{1, 2, \ldots, n\}$. Therefore, $x \in \bigcap_{j=1}^{n} (E\backslash A_j)$, showing that $E\backslash \bigcup_{j=1}^{n} A_j \subset \bigcap_{j=1}^{n} (E\backslash A_j)$. Conversely, suppose $x \in \bigcap_{j=1}^{n} (E\backslash A_j)$. Then $x \in E\backslash A_{j_0}$ for all $j \in \{1, 2, \ldots, n\}$. Thus $x \in E$ and $x \notin A_j$ for all $j \in \{1, 2, \ldots, n\}$. This implies $x \in E$ and $x \notin \bigcup_{j=1}^{n} A_j$. It follows that $x \in E\backslash \bigcup_{j=1}^{n} A_j$, so that $\bigcap_{j=1}^{n} (E\backslash A_j) \subset E\backslash \bigcup_{j=1}^{n} A_j$. Since the sets $E\backslash \bigcup_{j=1}^{n} A_j$ and $\bigcap_{j=1}^{n} (E\backslash A_j)$ contains the same elements, they are equal by Definition 1.1.

CHAPTER 2

The Real Numbers

Exercise (2.C.). Consider the subset of $\mathbf{R}\times\mathbf{R}$ defined by $D = \{(x, y) : |x| + |y| = 1\}$. Describe this set in words. Is it a function?

Solution. D is the set of all points lying on the edges of the square with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$ in $\mathbf{R}^2$.

We claim that D is not a function. For $-1 < x < 1$, both $(x, 1 - |x|)$ and $(x, |x| - 1)$ are in D, but $1 - |x| \neq |x| - 1$.

Exercise (2.E.). Prove that if f is an injection from A to B, then $f^{-1} = \{(b, a) : (a, b) \in f\}$is a function. Then prove it is an injection.

Solution. Suppose f^{-1} is not a function. Then for some $(b, a) \in f^{-1}$, there exists $(b, a') \in f^{-1}$, where $a \neq a'$. This implies $(a', b) \in f$. Hence both (a, b) and $(a'b)$ are in f, which contradicts the fact that f is an injection. Therefore, f is a function.

Suppose f^{-1} is not an injection. Then for some $(b, a) \in f^{-1}$, there exists $(b', a) \in f^{-1}$, where $b \neq b'$. This implies $(a, b') \in f$. Hence both (a, b) and (a, b') are in f, which contradicts the fact that f is a function. Therefore, f^{-1} is an injection.

Exercise (2.G.). Let f and g be functions and suppose that $g \circ f(x) = x$ for all x in $D(f)$. Show that f is injection and that $R(f) \subseteq D(g)$ and $R(g) \subseteq D(f)$.

Solution. Let $x_1, x_2 \in D(f)$ and suppose $f(x_1) = f(x_2)$. Then $g \circ f(x_1) = g \circ f(x_2)$, so $x_1 = x_2$. Thus f is an injection.

Let $y \in R(f)$, there exists $x \in D(f)$ such that $y = f(x)$. Since $g \circ f(x) = g(y) = x$, it follows that $y \in D(g)$, and hence $R(f) \subseteq D(g)$.

Let $u \in R(g)$, we have $g \circ f(u) = u$. This implies $f(u)$ is defined, so $u \in D(f)$, and hence $R(g) \subseteq D(f)$.

EXERCISE (2.H.). Let f, g be functions such that

$$g \circ f(x) = x, \qquad \text{for all } x \text{ in } D(f),$$
$$f \circ g(y) = y, \qquad \text{for all } y \text{ in } D(g),$$

Prove that $g = f^{-1}$.

SOLUTION. Apply Problem 2.G to $g \circ f(x) = x$, we obtain f is an injection. There exists f^{-1}. Apply Problem 2.G to $f \circ g(y) = y$, we obtain g is an injection. This implies $f^{-1} \circ f(x) = x = g \circ f(x)$, $\forall x \in D(f)$. It follows that $f^{-1}(y) = g(y)$, $\forall y \in R(f)$. Therefore, $g = f^{-1}$.

EXERCISE (2.J.). Let f be the function on $\mathbf{R}$ to $\mathbf{R}$ given by $f(x) = x^2$, and let $E = \{x \in \mathbf{R} : -1 \le x \le 0\}$ and $F = \{x \in R : 0 \le x \le 1\}$. Then $E \cap F = \{0\}$ and $f(E \cap F) = \{0\}$ while $f(E) = f(F) = \{y \in \mathbf{R} : 0 \le x \le 1\}$. Hence $f(E \cap F)$ is a proper subset of $f(E) \cap f(F)$. Now delete 0 from E and F.

SOLUTION. First, since $E \cap F = \{0\}$, so $f(E \cap F) = \{f(x) = x^2 : x = 0\} = \{0\}$. Moreover, since $f(E) = \{f(x) = x^2 : -1 \le x \le 0\} = [0, 1]$ and $f(F) = \{f(x) = x^2 : 0 \le x \le 1\} = [0, 1]$, so $f(E) \cap f(F) = [0, 1]$. Thus $f(E \cap F) \subsetneq f(E) \cap f(F)$.

If x is deleted from E and F, then $E \cap F = \emptyset$ and therefore $f(E \cap F) = \emptyset$. Moreover, $f(E) = f(F) = (0, 1]$.

CHAPTER 3

Finite and Infinite Sets

Exercise (3.B.). Exhibit a one-one correspondence between the set O of odd natural numbers and $\mathbf{N}$.

Solution. Define a correspondence $f\colon O \to \mathbf{N}$ as

$$f(n) = \frac{n+1}{2}$$

for all $n \in \mathbf{N}$. We will show that f is bijective.

To see that f is injective, we must show that $f(n) = f(m)$ implies $n = m$. Suppose $f(n) = f(m)$. This means $\frac{n+1}{2} = \frac{m+1}{2}$. It follows that $n = m$. Thus f is injective.

To see that f is surjective, we must show that for any element $n \in \mathbf{N}$, there is a corresponding element $o \in O$ for which $f(o) = n$. Consider an arbitrary $n \in \mathbf{N}$. Because $f(2n-1) = \frac{(2n-1)+1}{2} = n$, there is an element $o = 2n-1 \in O$ for which $f(o) = n$. Thus f is surjective.

Exercise (3.D.). If A is contained in some initial segment of $\mathbf{N}$, use the well-ordering property of $\mathbf{N}$ to define a bijection of A onto some initial segment of $\mathbf{N}$.

Solution. $A = \{a_1, a_2, \dots, a_k\}$ for $a_i \in \{1, 2, \dots, n\}$ for some $n \in \mathbf{N}$. WLOG, we assume that $a_1 < a_2 < \cdots < a_k$, by the well-ordering property of $\mathbf{N}$.

Conisder the initial segment $\{1, 2, \dots, k\}$ of $\mathbf{N}$ and define the maping $f\colon A \to \{1, 2, \dots, k\}$ as $f(a_i) = i$ for all $a_i \in A$.

It is easy to show that f is bijective.

Exercise (3.F.). Use the fact that every infinite set has a denumerable subset to show that every infinte set can be put into one-one correspondence with a proper subset of itself.

Solution. Let S is an infinite set and consider the denumerable subset $A = \{a_1, a_2, \dots, a_n, \dots\}$ of S. Since every infinite set has a denumerable subset, such set A exists. Let $A_1 = \{a_1, a_3, a_5, \dots\}$ and $A_2 =$

$\{a_2, a_4, a_6, \ldots\}$ be proper subsets of A, and hence proper subsets of S. Thus $S \backslash A_2$ is a proper subset of S. Define the mapping $f\colon S \to S \backslash A_2$ as

$$f(x) = \begin{cases} a_{2i-1} & \text{for } x = a_i \in A \\ x & \text{for } x \in S \backslash A \end{cases}$$

We claim that f is a one-one correpondence between S and $S \backslash A_2$.

To see that f is injective, we must show that $f(x) = f(y)$ implies $x = y$. Clearly, the restriction of f to $S \backslash A$ is injective. Suppose $f(a_i) = f(a_j)$ where $a_i, a_j \in A$. This means $a_{2i-1} = a_{2j-1}$. It follows that $a_i = a_j$, and hence the restriction of f to A is injective. Thus f is injective.

To see that f is surjective, we must show that for any element $y \in S \backslash A_2$, there is a corresponding element $x \in S$ for which $f(x) = y$. Clearly, the restriction of f to $S \backslash A$ is surjective.Consider an arbitrary $a_{2i-1} \in S \backslash A_2$. Because $f(a_i) = a_{2i-1}$, there is an element $x = a_i \in A \subset S$ for which $f(x) = y$, and hence the restriciton of f to A is surjective. Thus f is surjective.

Exercise (3.H.). Show that if the set A can be put into one-one correspondence with a set B, and if B can be put into one-one correspondence with a set C, then A can be put into one-one correspondence with C.

Solution. If the set A can be put into one-one correspondence with a set B, and if B can be put into one-one correspondence with a set C, then there exists one-one corresepondeces $f\colon A \to B$ and $g\colon B \to C$. We claim that the correspondence $g \circ f\colon A \to C$ is a one-one correspondece since the composition of one-one correspondences is an one-one corresponce.

Suppose both f and g are injective. To see that $g \circ f$ is injective, we must show that $g \circ f(x) = g \circ f(y)$ implies $x = y$. Suppose $g \circ f(x) = g \circ f(y)$. This means $g(f(x)) = g(f(y))$. It follows that $f(x) = f(y)$. For otherwise g wouldn't be injective.) But since $f(x) = f(y)$ and f is injective, it must be that $x = y$. Therefore $g \circ f$ is injective.

Suppose both f and g are surjective. To see that $g \circ f$ is surjective, we must show that for any element $c \in C$, there is a corresponding element $a \in A$ for which $g \circ f(a) = c$. Consider an arbitrary $c \in C$. Because g is surjective, there is an element $b \in B$ for which $g(b) = c$. And because f is surjective, there is an element $a \in A$ for which $f(a) = b$. Therefore $g(f(a)) = g(b) = c$, which means $g \circ f(a) = c$. Thus $g \circ f$ is surjective.

Exercise (3.I.). Using induction on $n \in \mathbf{N}$, show that the initial segment determines by n cannot be put into one-one correspondence with the inital segment determined by $m \in \mathbf{N}$, if $m < n$.

Solution. We use mathematical induction. If $n = 2$, then $m = 1$. Consider $A_2 = \{1, 2\}$ and $B_1 = \{1\}$. Assume there exists a bijection $f \colon A_2 \to B_1$, then $|A_2| = |B_1|$, a contradiction to the fact that $|A_2| = 2 \neq 1 = |B_1|$. Thus the assertion with $n = 2$ and $m = 1$ has just been proved.

Assume the assertion is true for $m < n = k$. That is, there is no bijection f from $A_k \to B_m$

Let $n = k + 1$, and assume that there exists a bijetion $f \colon A_{k+1} \to B_m$, then $f|_{A_k}$ is also a bijection from $A_k \to B_m \backslash \{f(k + 1)\}$, a contradiction to the inductive hypothesis.

CHAPTER 4

The Algebraic Properties of R

Exercise (4.C.). Prove part (b) of Theorem 4.4.

Solution. Since $a \cdot ((1/a) \cdot b) = (a \cdot (1/a)) \cdot b = 1 \cdot b = b$, it is clear that $x = (1/a) \cdot b$ is a solution of the equation $a \cdot x = b$. To establish that it is the only solution, let x_1 be any solution of this equation; hence

$$a \cdot x_1 \quad = \quad b.$$

We multiply $(1/a)$ to both sides to obtain

$$(1/a) \cdot (a \cdot x_1) \quad = \quad (1/a) \cdot b.$$

If we employ (M3), (M4), and (M2), we get

$$\begin{aligned} x_1 &= (1/a) \cdot b \\ 1 \cdot x_1 &= (1/a) \cdot b \\ ((1/a) \cdot a) \cdot x_1 &= (1/a) \cdot b \\ (1/a) \cdot (a \cdot x_1) &= (1/a) \cdot b \end{aligned}$$

Hence $x_1 = (1/a) \cdot b$.

Exercise (4.F.). Use the argument in Theorem 4.7 to show that there does not exist a rational number s such thatt $s^2 = 6$.

Solution. Suppose, on the contrary that $(p/q)^2 = 6$, where p and q are integers. We may, without loss of generality, suppose that p and q have no common integral factors. Since $p^2 = 6q^2$, it follows that p^2 must be a multiple of 6. We may classify p in the following cases,

$$
\begin{aligned}
p &= 6k, \\
p &= 6k+1, \\
p &= 6k+2, \\
p &= 6k+3, \\
p &= 6k+4, \\
p &= 6k+5.
\end{aligned}
$$

Thus

$$
\begin{aligned}
p^2 &= 36k^2, \\
p^2 &= 36k^2+12k+1, \\
p^2 &= 36k^2+24k+4, \\
p^2 &= 36k^2+36k+9, \\
p^2 &= 36k^2+48k+16, \\
p^2 &= 36k^2+60k+25.
\end{aligned}
$$

Therefore $p = 6k$ for some integer k and hence $36k^2 = 6q^2$ (for if p is in one of the latter five cases, p^2 is not a multiple of 6). It follows that $q^2 = 6k^2$, whence q must also be a multiple of 6 by the reasoning above. Therefore both p and q are divisible by 6, contrary to our hypothesis.

Exercise (4.G.). Modify the argument in Theorem 4.7 to show that there does not exists a rational number t such that $t^2 = 3$.

Solution. Suppose, on the contrary that $(p/q)^2 = 3$, where p and q are integers. We may, without loss of generality, suppose that p and q have no common integral factors. Since $p^2 = 3q^2$, it follows that p^2 must be a multiple of 3. We may classify p in the following cases,

$$
\begin{aligned}
p &= 3k, \\
p &= 3k+1, \\
p &= 3k+2.
\end{aligned}
$$

Thus

$$\begin{aligned} p^2 &= 9k^2, \\ p^2 &= 9k^2 + 6k + 1, \\ p^2 &= 9k^2 + 12k + 4. \end{aligned}$$

Therefore $p = 3k$ for some integer k and hence $9k^2 = 3q^2$ (for if p is in one of the latter two cases, p^2 is not a multiple of 3). It follows that $q^2 = 3k^2$, whence q must also be a multiple of 3 by the reasoning above. Therefore both p and q are divisible by 3, contrary to our hypothesis.

Exercise (4.H.). If $\xi \in \mathbf{R}$ is irrational and $r \in \mathbf{R}$, $r \neq 0$, is rational, show that $r + \xi$ and $r\xi$ are irrational.

Solution. Given that ξ is irrational. We are also given that r is rational, $r \neq 0$, we can write $r = a/b$, where a, $a \neq 0$, and b are integers. We may, without loss of generality, suppose that a and b have no common integral factors.

Suppose, on the contrary that $r + \xi = c/d$, where c, and d are integers. We may again, without loss of generality, suppose that c and d have no common integral factors. Since

$$\begin{aligned} \xi &= \frac{c}{d} - r \\ &= \frac{c}{d} - \frac{a}{b} \\ &= \frac{bc - ad}{bd}, \end{aligned}$$

it follows that ξ is rational, contrary to our hypothesis.

Suppose, on the contrary that $r\xi = c/d$, where c, and d are integers. We may again, without loss of generality, suppose that c and d have no common integral factors. Since

$$\begin{aligned} \xi &= \frac{c}{rd} \\ &= \frac{c}{\frac{a}{b}d} \\ &= \frac{bc}{ad}, \end{aligned}$$

it follows that ξ is rational, contrary to our hypothesis.

CHAPTER 5

The Order Properties of R

EXERCISE (5.B.). If $n \in \mathbf{N}$, show that $n^2 \geq n$ and hence$1/n^2 \leq 1/n$.

SOLUTION. We first show that

$$n \leq n^2$$

for all $n \in \mathbf{N}$. The proof uses Mathematical Induction. If $n = 1$, $1 = 1^2$. The assertion with $n = 1$ has just been proved. Supposing the assertion true for the natural number $k > 1$. That is, supposing $k \leq k^2$. Then, since

$$\begin{aligned} k + 1 &< k^2 + 1 \\ &< k^2 + 2k + 1, \end{aligned}$$

it follows that $k + 1 < (k + 1)^2$.

We next show that

$$1/n^2 \leq 1/n$$

for all $n \in \mathbf{N}$. The proof uses Mathematical Induction. If $n = 1$, $1/1^2 = 1 = 1/1$. The assertion with $n = 1$ has just been proved. Supposing the assertion true for the natural number $k > 1$. That is, supposing $1/k^2 < 1/k$. Then, since

$$\begin{aligned} \frac{1}{k+1} - \frac{1}{(k+1)^2} &= \frac{k}{(k+1)^2} \\ &> 0, \end{aligned}$$

for $k > 1$, it follows that $1/(k + 1)^2 < 1/(k + 1)$.

EXERCISE (5.C.). If $a > -1$, $a \in \mathbf{R}$, show that $(1 + a)^n \geq 1 + na$ for all $n \in \mathbf{N}$. This inequality is called **Bernoulli's Inequality**.

SOLUTION. The proof uses Mathematical Induction. If $n = 1$, $(1+a)^1 = 1 + 1a$. The assertion with $n = 1$ has just been proved. Supposing the assertion true for the natural number $k > 1$. That is, supposing $(1+a)^k \geq 1 + ka$. Then, since

$$\begin{aligned} (1+a)^{k+1} &= (1+a)^k(1+a) \\ &\geq (1+ka)(a+a) \\ &= 1 + (k+1)a + \underbrace{kx^2}_{\geq 0}, \end{aligned}$$

it follows that $(1+a)^{k+1} \geq 1 + (k+1)a$.

EXERCISE (5.F.). Suppose $0 < c < 1$. If $m \geq n$, $m, n \in \mathbf{N}$, show that $0 < c^m \leq c^n < 1$.

SOLUTION. We first show that

$$c^n \leq c$$

for all $0 < c < 1$ and $n \in \mathbf{N}$. The proof uses Mathematical Induction. If $n = 1$, $c^1 = c$. If $n = 2$, $c^2 < c$ as $c - c^2 = c(1-c) > 0$ (for $c > 0$ and $1 - c > 0$). The assertion with $n = 1$ and $n = 2$ has just been proved. Supposing the assertion true for the natural number k. That is, supposing $c^k \leq c$. Then, since

$$\begin{aligned} c^{k+1} &= c^k c \\ &\leq c^2 \\ &< c, \end{aligned}$$

it follows that $c^{k+1} < c < 1$.

We next show that

$$c^m \leq c^n$$

for all $0 < c < 1$ and $m \geq n$, $m, n \in \mathbf{N}$. The proof uses Mathematical Induction. If $m = n$, $c^m = c^n$. If $m = n + 1$, $c^m < c^n$ as $c^n - c^m = c^n(1-c) > 0$ (for c and $1 - c > 0$). The assertion with $m = n$ and $m = n + 1$ has just been proved. Supposing the assertion true for the natural number $m = n + k$, where $k \geq 2$. That is, supposing $c^{n+k} \leq c^n$, where $k \geq 2$. Then, since

$$\begin{aligned} c^{n+k+1} &= c^{n+k}c \\ &\leq c^{n+1}, \end{aligned}$$

it follows that $c^{n+k+1} < c^n$.

Moreover, $c^m > 0$ for all $0 < c < 1$ and $m \in \mathbf{N}$. Therefore $0 < c^m \leq c^n < 1$ for all $0 < c < 1$ and $m \geq n$, $m, n \in \mathbf{N}$.

Exercise (5.G.). Show that $n < 2^n$ for all $n \in \mathbf{N}$. Hence $1/2^n < 1/n$ for all $n \in \mathbf{N}$.

Solution. We first show that

$$n < 2^n$$

for all $n \in \mathbf{N}$. The proof uses Mathematical Induction. If $n = 1$, $1 < 2 = 2^1$. The assertion with $n = 1$ has just been proved. Supposing the assertion true for the natural number $k > 1$. That is, supposing $k < 2^k$. Then, since

$$\begin{aligned} k+1 &< 2^k + 1 \\ &< 2^k + 2, \end{aligned}$$

it follows that $k + 1 < 2^{k+1}$.

We next show that

$$1/2^n < 1/n$$

for all $n \in \mathbf{N}$. The proof uses Mathematical Induction. If $n = 1$, $1/2^1 = 1/2 < 1 = 1/1$. The assertion with $n = 1$ has just been proved. Supposing the assertion true for the natural number $k > 1$. That is, supposing $1/2^k < 1/k$. Then, since

$$\begin{aligned} \frac{1}{2^{k+1}} &= \frac{1}{2^k}\frac{1}{2} \\ &< \frac{1}{k}\frac{1}{2} \\ &= \frac{1}{2k}, \end{aligned}$$

and $1/2k < 1/(k+1)$ as

$$\frac{1}{k+1} - \frac{1}{2k} = \frac{k-1}{2k(k+1)} > 0$$

for $k > 1$, it follows that $1/2^{k+1} < 1/(k+1)$.

EXERCISE (5.K.). If $a, b \in \mathbf{R}$ and $b \neq 0$, show that $|a/b| = |a| / |b|$.

SOLUTION. If $a > 0$ and $1/b > 0$, then $a/b > 0$ so that $|a/b| = a/b = |a| / |b|$. If $a < 0$ and $1/b > 0$, then $a/b < 0$ so that $|ab| = -(a/b) = (-a)/b = |a| / |b|$. The other cases are handled similarly. Specifically, if $a < 0$ and $1/b < 0$, then $a/b > 0$ so that $|a/b| = a/b = |a| / |b|$. If $a > 0$ and $1/b < 0$, then $a/b < 0$ so that $|ab| = -(a/b) = a/(-b) = |a| / |b|$.

EXERCISE (5.L.). Show that $a, b \in \mathbf{R}$, then $|a+b| = |a| + |b|$ if and only if $ab \geq 0$.

SOLUTION. Suppose $|a+b| = |a| + |b|$ for $a, b \in \mathbf{R}$. Then

$$\begin{aligned} |a+b|^2 &= (|a| + |b|)^2 \\ (a+b)^2 &= |a|^2 + |b|^2 + 2|a||b| \\ a^2 + b^2 + 2ab &= |a|^2 + |b|^2 + 2|a||b|. \end{aligned}$$

Moreover, we shall show that $a^2 = |a|^2$ for all $a \in \mathbf{R}$. Indeed, since $a^2 \geq 0$, we have $a^2 = \left|a^2\right| = |aa| = |a||a| = |a|^2$. Thus, since

$$\begin{aligned} a^2 + b^2 + 2ab &= a^2 + b^2 + 2|a||b| \\ 2ab &= 2|ab| \end{aligned}$$

it follows that $ab = |ab|$. Suppose, on the contrary that $ab < 0$. Then $|ab| = -ab$. This implies $ab = -ab$, that is, $2ab = 0$, so that $ab = 0$, contrary to our hypothesis. Therefore $ab \geq 0$.

Conversely, suppose $ab \geq 0$ for $a, b \in \mathbf{R}$. Then either $ab = 0$ or $ab > 0$. If $ab = 0$, then either $a = 0$ or $b = 0$. We may , without loss of generality, suppose that $a = 0$. Then $|a+b| = |0+b| = |b| = |0+b| = |a| + |b|$. If $ab > 0$, then either $a > 0, b > 0$ or $a < 0, b < 0$. If $a > 0, b > 0$, then $a+b > 0$ so that $|a+b| = a+b = |a| + |b|$. If $a < 0, b < 0$, then $a+b < 0$ so that $|a+b| = -(a+b) = -a+(-b) = |a| + |b|$. Therefore $|a+b| = |a| + |b|$.

CHAPTER 6

The Completeness Property of $\mathbf{R}$

Exercise (6.B.). If a subset S of $\mathbf{R}$ contains an upper bound, then this upper bound is the supremum of S.

Solution. Let $u \in S$ is an upper bound of S. Then $s \leq u$ for all $s \in S$. Suppose, on the contrary that u is not the supremum of S. Then there exists $v \in S$, $v < u$, is the supremum of S, by Supremum Property. Therefore $s \leq v$ for all $s \in S$, it follows that $u < v$ (for $u \in S$) and $u \neq v$ (for, if $u = v$, then u is the supremum of S), contradicting our hypothesis.

Exercise (6.C.). Give an example of a set of rational numbers which is bounded but which does not have a rational supremum.

Solution. A typical of a set of rational numbers which is bounded but which does not have a rational supremum is obtained by defining

$$\mathbf{Q}_{[0,\sqrt{2}]} = \{x \in \mathbf{Q} : 0 \leq x \leq \sqrt{2}\}.$$

The set $\mathbf{Q}_{[0,\sqrt{2}]}$ is bounded and $\sup \mathbf{Q}_{[0,\sqrt{2}]} = \sqrt{2} \notin \mathbf{Q}$.

Exercise (6.D.). Give an example of a set of irrational numbers which has a rational supremum.

Solution. A typical of a set of irrational numbers which is bounded but which has a rational supremum is obtained by defining

$$\mathbf{I}_{[0,1]} = \{x \in \mathbf{I} : 0 \leq x \leq 1\}.$$

The set $\mathbf{I}_{[0,1]}$ is bounded and $\sup \mathbf{I}_{[0,1]} = 1 \in \mathbf{Q}$.

Exercise (6.G.). If S is a bounded set in $\mathbf{R}$ and if S_0 is a nonempty subset of S, then show that

$$\inf S \leq \inf S_0 \leq \sup S_0 \leq \sup S.$$

Sometimes it is more convenient to express this in another way. Let $D \neq \emptyset$ and let $f\colon D \to \mathbf{R}$ have bounded range. If D_0 is a nonempty subset of D, then

$$\inf\{f(x)\colon x \in D\} \leq \inf\{f(x) : x \in D_0\} \leq \sup\{f(x) : x \in D_0\} \leq \sup\{f(x) : x \in D\}.$$

Solution. We first show that $\inf S \leq \inf S_0$. For all $t \in S_0$, then $t \in S$, implying that $\inf S \leq t$. Thus $\inf S$ is a lower bound for S_0 and the ineqality follows.

We next show that $\inf S_0 \leq \sup S_0$. For all $t \in S_0$, $\inf S_0 \leq t \leq \sup S_0$, by the definitons of infimum and supremum, and the ineqality follows.

We finally show that $\sup S_0 \leq \sup S$. For all $t \in S_0$, then $t \in S$, implying that $\sup t \geq t$. Thus $\sup S$ is an upper bound for S_0 and the inequality follows.

Exercise (6.H.). Let X and Y be non-empty sets and let $f\colon X \times Y \to \mathbf{R}$ have bounded range. Let

$$\begin{aligned} f_1(x) &= \sup\{f(x,y) : y \in Y\}, \\ f_2(x) &= \sup\{f(x,y)\colon x \in X\}. \end{aligned}$$

Establish the **Property of Iterated Suprema**:

$$\begin{aligned} \sup\{f(x,y) : x \in X, y \in Y\} &= \sup\{f_1(x) : x \in X\} \\ &= \sup\{f_2(y) : y \in Y\}. \end{aligned}$$

We sometimes express this in symbols by

$$\sup_{x,y} f(x,y) = \sup_x \sup_y f(x,y) = \sup_y \sup_x f(x,y).$$

Solution. Let $\gamma = \sup\{f_1(x) : x \in X\}$. We shall prove that γ is the supremum of $\{f(x,y) : x \in X, y \in Y\}$. Since

$$\begin{aligned} f_1(x) &\leq \gamma \\ \sup\{f(x,y) : y \in Y\} &\leq \gamma, \end{aligned}$$

for all $x \in X$, it follows that $f(x,y) \leq \sup\{f(x,y) : y \in Y\} \leq \gamma$ for all $(x,y) \in X \times Y$. Thus γ is an upper bound of the set $\{f(x,y) : x \in X, y \in Y\}$. Hence by the Supemum Property, the supremum of $\{f(x,y) : x \in X, y \in Y\}$ exists, and $\sup\{f(x,y) : x \in X, y \in Y\} \leq \gamma$. We need to show that γ is the supremum of $\{f(x,y) : x \in X, y \in Y\}$.

For a given $\epsilon > 0$, since $\gamma = \sup\{f_1(x) : x \in X\}$, so there exists $x_0 \in X$ such that $f_1(x_0) > \gamma - \epsilon$ (for, otherwise, $\gamma - \epsilon = \sup\{f_1(x) : x \in X\}$),

that is, $\sup\{f(x_0, y) : y \in Y\} > \gamma - \epsilon$. It follows that there exists $y_0 \in Y$ such that $f(x_0, y_0) > \gamma - \epsilon$ (for, otherwise, $\gamma - \epsilon = \sup\{f(x_0, y) : y \in Y\}$), implying $\gamma - \epsilon \neq \sup\{f(x, y) : x \in X, y \in Y\}$. Since this is valid for all ϵ, so there does not exist an $\epsilon > 0$ such that $\gamma - \epsilon$ is the supremum of $\{f(x, y) : x \in X, y \in Y\}$, showing that γ is the supremum of $\{f(x, y) : x \in X, y \in Y\}$. Therefore $\sup\{f(x, y) : x \in X, y \in Y\} = \sup\{f_1(x) : x \in X\}$.

Let $\delta = \sup\{f_2(x) : y \in Y\}$. We shall prove that δ is the supremum of $\{f(x, y) : x \in X, y \in Y\}$. Since

$$\begin{aligned} f_2(x) &\leq \delta \\ \sup\{f(x, y) : x \in X\} &\leq \delta, \end{aligned}$$

for all $y \in Y$, it follows that $f(x, y) \leq \sup\{f(x, y) : x \in X\} \leq \delta$ for all $(x, y) \in X \times Y$. Thus δ is an upper bound of the set $\{f(x, y) : x \in X, y \in Y\}$. Hence by the Supemum Property, the supremum of $\{f(x, y) : x \in X, y \in Y\}$ exists, and $\sup\{f(x, y) : x \in X, y \in Y\} \leq \delta$. We need to show that δ is the supremum of $\{f(x, y) : x \in X, y \in Y\}$.

For a given $\epsilon > 0$, since $\delta = \sup\{f_2(x) : y \in Y\}$, so there exists $y_0 \in Y$ such that $f_2(y_0) > \delta - \epsilon$ (for, otherwise, $\delta - \epsilon = \sup\{f_2(y) : y \in Y\}$), that is, $\sup\{f(x, y_0) : x \in X\} > \gamma - \epsilon$. It follows that there exists $x_0 \in X$ such that $f(x_0, y_0) > \gamma - \epsilon$ (for, otherwise, $\delta - \epsilon = \sup\{f(x, y_0) : x \in X\}$), implying $\delta - \epsilon \neq \sup\{f(x, y) : x \in X, y \in Y\}$. Since this is valid for all ϵ, so there does not exist an $\epsilon > 0$ such that $\delta - \epsilon$ is the supremum of $\{f(x, y) : x \in X, y \in Y\}$, showing that δ is the supremum of $\{f(x, y) : x \in X, y \in Y\}$. Therefore $\sup\{f(x, y) : x \in X, y \in Y\} = \sup\{f_2(y) : y \in Y\}$.

Exercise (6.J.). Let X be a non-empty set and let $f : X \to \mathbf{R}$ have bounded range in $\mathbf{R}$. If $a \in \mathbf{R}$, show that

$$\begin{aligned} \sup\{a + f(x) : x \in X\} &= a + \sup\{f(x) : x \in X\}, \\ \inf\{a + f(x) : x \in X\} &= a + \inf\{f(x) : x \in X\}. \end{aligned}$$

Solution. Since

$$f(x) \leq \sup\{f(x) : x \in X\}$$

for all $x \in X$, implying that

$$a + f(x) \leq a + \sup\{f(x) : x \in X\}$$

for all $x \in X$, so that

$$a + \sup\{f(x) : x \in X\}$$

is an upper bound of $a + f(x)$ for all $x \in X$. Thus, by the Supemum Property, the supremum of $\{a + f(x) : x \in X\}$ exists, and

$$\sup\{a + f(x) : x \in X\} \leq a + \sup\{f(x) : x \in X\}.$$

On the other hand, since

$$a + f(x) \leq \sup\{a + f(x) : x \in X\}$$

for all $x \in X$, implying that

$$f(x) \leq \sup\{a + f(x) : x \in X\} - a$$

for all $x \in X$, so that

$$\sup\{a + f(x) : x \in X\} - a$$

is an upper bound of $f(x)$. Thus, by the Supemum Property, the supremum of $\{f(x) : x \in X\}$ exists, and

$$\sup\{f(x) : x \in X\} \leq \sup\{f(x) : x \in X\} - a,$$

that is,

$$a + \sup\{f(x) : x \in X\} \leq \sup\{f(x) : x \in X\}.$$

Therefore,

$$\sup\{a + f(x) : x \in X\} = a + \sup\{f(x) : x \in X\}.$$

Since

$$f(x) \geq \sup\{f(x) : x \in X\}$$

for all $x \in X$, implying that

$$a + f(x) \geq a + \inf\{f(x) : x \in X\}$$

for all $x \in X$, so that

$$a + \inf\{f(x) : x \in X\}$$

is a lower bound of $a + f(x)$ for all $x \in X$. Thus, by the Infimum Property, the infimum of $\{a + f(x) : x \in X\}$ exists, and

$$\inf\{a + f(x) : x \in X\} \geq a + \inf\{f(x) : x \in X\}.$$

On the other hand, since

$$a + f(x) \geq \inf\{a + f(x) : x \in X\}$$

for all $x \in X$, implying that

$$f(x) \geq \inf\{a + f(x) : x \in X\} - a$$

for all $x \in X$, so that

$$\inf\{a + f(x) : x \in X\} - a$$

is a lower bound of $f(x)$. Thus, by the Infimum Property, the infimum of $\{f(x) : x \in X\}$ exists, and

$$\inf\{f(x) : x \in X\} \geq \inf\{f(x) : x \in X\} - a,$$

that is,

$$a + \inf\{f(x) : x \in X\} \geq \inf\{f(x) : x \in X\}.$$

Therefore,

$$\inf\{a + f(x) : x \in X\} = a + \inf\{f(x) : x \in X\}.$$

EXERCISE (6.K.). Let X be a non-empty set and let f and g ve defined on X and have bounded ranges in $\mathbf{R}$. Show that

$$\begin{aligned}
\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} &\leq \inf\{f(x) + g(x) : x \in X\} \\
&\leq \inf\{f(x) : x \in X\} + \sup\{g(x) : x \in X\} \\
&\leq \sup\{f(x) + g(x) : x \in X\} \\
&\leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}.
\end{aligned}$$

Give examples to show that each inequality can be strict.

SOLUTION. We first show that

$$\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} \leq \inf\{f(x) + g(x) : x \in X\}.$$

Since $\inf\{f(x) : x \in X\} \leq f(x)$ and $\inf\{g(x) : x \in X\} \leq g(x)$ for all $x \in X$, it follows that

$$\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} \leq f(x) + g(x)$$

for all $x \in X$. Hence

$$\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\}$$

is a lower bound of $f(x)+g(x)$. Thus, by the Infimum Property, the infimum of $\{f(x) + g(x) : x \in X\}$ exists, and

$$\inf\{f(x) + g(x) : x \in X\} \geq \inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\}.$$

We next show that

$$\inf\{f(x) + g(x) : x \in X\} \leq \inf\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}.$$

Since $\inf\{f(x) + g(x) : x \in X\} \leq f(x) + g(x)$ for all $x \in X$, it follows that

$$\inf\{f(x) + g(x) : x \in X\} \leq f(x) + \sup(g(x)).$$

Taking infimum both sides and applying Problem 6.J to this inequality, we have

$$\inf(\inf\{f(x)+g(x) : x \in X\}) \leq \inf(f(x)+\sup(g(x))),$$

so that

$$\inf\{f(x)+g(x) : x \in X\} \leq \inf\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}.$$

We then show that

$$\inf\{f(x) : x \in X\} + \sup\{g(x) : x \in X\} \leq \sup\{f(x)+g(x) : x \in X\}.$$

Since

$$\inf\{f(x) : x \in X\} + g(x) \leq f(x)+g(x)$$

for all $x \in X$, taking supreumum both sides of this inequality, it follows that

$$\sup(\inf\{f(x) : x \in X\} + g(x)) \leq \sup\{f(x)+g(x) : x \in X\},$$

for all $x \in X$. Applyting Problem 6.J to this inequality, we obtain

$$\inf\{f(x) : x \in X\} + \sup\{g(x) : x \in X\} \leq \sup\{f(x)+g(x) : x \in X\}.$$

We finally show that

$$\sup\{f(x)+g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

Since

$$f(x)+g(x) \leq \sup(f(x)) + \sup(g(x))$$

for all $x \in X$, it follows that $\sup(f(x)) + \sup(g(x))$ is an upper bound of $f(x)+g(x)$. Thus, by the Supremum Property, the supremum of $\{f(x)+g(x) : x \in X\}$ exists, and

$$\sup\{f(x)+g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}.$$

CHAPTER 7

Cuts, Intervals, and the Cantor Set

Exercise (7.E.). Let $I_n = (n, +\infty)$ for $n \in \mathbf{N}$. Show that this sequence of intervals is nested, but that there is no common point.

Solution. Suppose, on the contrary that $\bigcap_{n=1}^{\infty} J_n \neq \emptyset$. Then there exists $x \in \bigcap_{n=1}^{\infty} J_n$. Thus $x \in (n, +\infty) = I_n$. By the Archimedian Property, there exists $K_x \in \mathbf{N}$ such that $x < K$. This implies $x \notin (K, +\infty) = I_K$, a contradiction. Therefore $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Exercise (7.F.). Let $J_n = (0, 1/n)$ for $n \in \mathbf{N}$. Show that this sequence of intervals is nested, but that there is no common point.

Solution. Suppose, on the contrary that $\bigcap_{n=1}^{\infty} J_n \neq \emptyset$. Then there exists $x \in \bigcap_{n=1}^{\infty} J_n$. Thus $x \in (0, 1) = J_1$. By the Archimedian Property, there exists $K \in \mathbf{N}$ such that $1/K < x$. This implies $x \notin (1, 1/K) = J_K$, a contradiction. Therefore $\bigcap_{n=1}^{\infty} J_n = \emptyset$.

Exercise (7.G.). If $I_n = [a_n, b_n]$, $n \in \mathbf{N}$, is a nested sequence of closed cells, show that

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq b_m \leq \cdots \leq b_2 \leq b_1.$$

If we put $\xi = \sup\{a_n : n \in \mathbf{N}\}$ and $\eta = \inf\{b_m : m \in \mathbf{N}\}$, show that $[\xi, \eta] = \bigcap_{n \in \mathbf{N}} I_n$.

Solution. The proof uses Mathematical Induction. If $n = 1$, $I_1 = [a_1, b_1]$, so that $a_1 \leq b_1$. The assertion with $n = 1$ has just been proved. Supposing the assertion true for the natural number $k > 1$. That is, supposing

$$a_1 \leq a_2 \leq \cdots \leq a_k \leq b_k \leq \cdots \leq b_2 \leq b_1.$$

Then, if $n = k + 1$, we need to show that

$$a_1 \leq a_2 \leq \cdots \leq a_k \leq a_{k+1} \leq b_{k+1} \leq b_k \leq \cdots \leq b_2 \leq b_1.$$

In order to do show, we first show that $a_k \leq a_{k+1}$ for all $k \in \mathbf{N}$. Since $a_{k+1} \in I_{k+1} \subseteq I_k$, it follows that $a_{k+1} \in I_k$, implying that $a_k \leq a_{k+1} \leq b_k$. Hence $a_k \leq a_{k+1}$. We next show that $b_k \geq b_{k+1}$ for all $k \in \mathbf{N}$. Since $b_{k+1} \in I_{k+1} \subseteq I_k$, it follows that $b_{k+1} \in I_k$, implying that $a_k \leq b_{k+1} \leq b_k$. Hence $b_k \geq b_{k+1}$. Thus $a_k \leq a_{k+1} \leq b_{k+1} \leq b_k$, showing that the assertion is true for $n = k + 1$.

Suppose $x \in [\xi, \eta]$, then $\xi \leq x \leq \eta$. Since $\xi = \sup\{a_n : n \in \mathbf{N}\}$, it follows that x is an upper bound of $\{a_n : n \in \mathbf{N}\}$, implying that $x \geq a_n$ for all $n \in \mathbf{N}$. Since $\eta = \inf\{b_m : m \in \mathbf{N}\}$, it follows that x is an lower bound of $\{b_n : n \in \mathbf{N}\}$, implying that $x \leq b_n$ for all $n \in N$. Thus, $a_n \leq x \leq b_n$ for all $n \in \mathbf{N}$, implying that $x \in [a_n, b_n]$ for all $n \in \mathbf{N}$. Therefore $x \in \bigcap_{n\in\mathbf{N}}[a_n, b_n] = \bigcap_{n\in\mathbf{N}} I_n$, so that $[\xi, \eta] \subset \bigcap_{n\in\mathbf{N}} I_n$. Conversely, suppose $x \in \bigcap_{n\in\mathbf{N}} I_n$, then $a_n \leq x \leq b_n$ for all $n \in \mathbf{N}$. Thus x is an upper bound of $\{a_n : n \in \mathbf{N}\}$ and a lower bound of $\{b_m : m \in \mathbf{N}\}$, implying that $x \geq \sup\{a_n : n \in \mathbf{N}\} = \xi$ and $x \leq \inf\{b_m : m \in \mathbf{N}\}$. Therefore $x \in [\xi, \eta]$, so that $\bigcap_{n\in\mathbf{N}} I_n \subset [\xi, \eta]$.

Exercise (7.K.). By removing sets with ever decreasing length, show that we can construct a "Cantor-like" set which has positive length. How large can we make the length of this set?

Solution. We consider the set of real numbers in $J = [0, 3]$. Firstly, if we remove the closed interval $[0, 1]$,we obtain the set $G_1 = (1, 3]$. Secondly, if we remove the closed interval $[1, \frac{4}{3}]$, we obtain the set $G_2 = (\frac{4}{3}, 3]$. In general, if we remove the closed interval $[\frac{2(n-1)}{n}, \frac{2n}{n+1}]$, we obtain the set $G_n = (\frac{2n}{n+1}, 3]$. Since $\frac{2n}{n+1} < 2$, for all $n \in \mathbf{N}$,it follows that $[2, 3] \subset G_n$ for all $n \in \mathbf{N}$. This set has the length of 1 after we remove infinitely many closed interval of the form $[\frac{2(n-1)}{n}, \frac{2n}{n+1}]$.

Part 2

The Topology of Cartesian Spaces

CHAPTER 8

Vector and Cartesian Spaces

Exercise (8.D.). If w_1 and w_2 are strictly positive, show that the definition

$$(x_1, x_2) \cdot (y_1, y_2) = x_1 y_1 w_1 + x_2 y_2 w_2,$$

yields an inner product on $\mathbf{R}^2$. Generalize this to $\mathbf{R}^p$.

Solution. (i) For all $x = (x_1, x_2) \in \mathbf{R}^2$ and w_1 and w_2 are strictly positive, we have

$$\begin{aligned}(x_1, x_2) \cdot (x_1, x_2) &= x_1 x_1 w_1 + x_2 x_2 w_2 \\ &= x_1^2 w_1 + x_2^2 w_2.\end{aligned}$$

Since $x_1^2, x_2^2 \geq 0$ and $w_1, w_2 > 0$, it follows that $x_1^2 w_1 + x_2^2 w_2 \geq 0$. Thus $(x_1, x_2) \cdot (x_1, x_2) \geq 0$ for all $x = (x_1, x_2) \in \mathbf{R}^2$, showing that the product defined above satisfies property (i) in 8.3.

(ii) We have

$$\begin{aligned}(x_1, x_2) \cdot (x_1, x_2) &= 0 \\ x_1 x_1 w_1 + x_2 x_2 w_2 &= 0 \\ x_1^2 w_1 + x_2^2 w_2 &= 0.\end{aligned}$$

Since $x_1^2, x_2^2 \geq 0$ and $w_1, w_2 > 0$, it follows that $x_1^2 w_1 + x_1^2 w_2 \geq 0$. The equality happens if and only if $x_1^2 w_1 = x_2^2 w_2 = 0$. Since $w_1, w_2 > 0$, so $x_1^2 w_1 = x_2^2 w_2 = 0$ if and only if $x_1 = x_2 = 0$, showing that the product defined above satisfies property (ii) in 8.3.

(iii) For all $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbf{R}^2$, since the ordinary multiplication in $\mathbf{R}$ is commute, so

$$\begin{aligned}(x_1, x_2) \cdot (y_1, y_2) &= x_1 y_1 w_1 + x_2 y_2 w_2 \\ &= y_1 x_1 w_1 + y_2 x_2 w_2 \\ &= (y_1, y_2)(x_1, x_2),\end{aligned}$$

showing that the product defined above satisfies property (iii) in 8.3.

(iv) For all $x = (x_1, x_2), y = (y_1, y_2)$, and $z = (z_1, z_2)$ in $\mathbf{R}^2$, since the ordinary multiplication and addition in $\mathbf{R}$ are commute and associative, so

$$\begin{aligned}(x_1, x_2) \cdot ((y_1, y_2) + (z_1, z_2)) &= (x_1, x_2) \cdot (y_1 + z_1, y_2 + z_2) \\ &= x_1(y_1 + z_1)w_1 + x_2(y_2 + z_2)w_2 \\ &= x_1 y_1 w_1 + x_1 z_1 w_1 + x_2 y_2 w_2 + x_2 z_2 w_2 \\ &= x_1 y_1 w_1 + x_2 y_2 w_2 + x_1 z_1 w_1 + x_2 z_2 w_2 \\ &= (x_1, x_2) \cdot (x_2, y_2) + (x_1, x_2) \cdot (z_1, z_2),\end{aligned}$$

and

$$\begin{aligned}((x_1, x_2) + (y_1, y_2)) \cdot (z_1, z_2) &= (x_1 + y_1, x_2 + y_2) \cdot (z_1, z_2) \\ &= (x_1 + y_1)z_1 w_1 + (x_2 + y_2)z_2 w_2 \\ &= x_1 z_1 w_1 + y_1 z_1 w_1 + x_2 z_2 w_2 + y_2 z_2 w_2 \\ &= x_1 z_1 w_1 + x_2 z_2 w_2 + y_1 z_1 w_1 + y_2 z_2 w_2 \\ &= (x_1, x_2) \cdot (z_1, z_2) + (y_1, y_2) \cdot (z_1, z_2),\end{aligned}$$

showing that the product defined above satisfies property (iv) in 8.3.

(v) For all $a \in \mathbf{R}$ and $x = (x_1, x_2)$, $y = (y_1, y_2)$ in $\mathbf{R}^2$, we have

$$\begin{aligned}(a(x_1, x_2)) \cdot (y_1, y_2) &= (ax_1, ax_2) \cdot (y_1, y_2) \\ &= ax_1 y_1 w_1 + ax_2 y_2 w_2 \\ &= a(x_1 y_1 w_1 + x_2 y_2 w_2) \\ &= a((x_1, x_2) \cdot (y_1, y_2)),\end{aligned}$$

and

$$\begin{aligned} a\left((x_1, x_2)\cdot(y_1, y_2)\right) &= a\left(x_1y_1w_1 + ax_2y_2w_2\right) \\ &= ax_1y_1w_1 + ax_2y_2w_2 \\ &= x_1(ay_1)w_1 + x_2(ay_2)w_2 \\ &= (x_1, x_2)\cdot(ay_1, ay_2) \\ &= (x_1, x_2)\cdot(a(y_1, y_2)), \end{aligned}$$

showing that the product defined above satisfies property (v) in 8.3.

In $\mathbf{R}^p$, we define

$$(x_1, x_2, \ldots, x_p)\cdot(y_1, y_2, \ldots, y_p) = x_1y_1w_1 + x_2y_2w_2 + \cdots + x_py_pw_p,$$

for all $x = (x_1, x_2, \ldots, x_p), y = (y_1, y_2, \ldots, y_p)$ in $\mathbf{R}^p$ and $w_1, w_2, \ldots, w_p > 0$ for all p.

Exercise (8.E.). The definition

$$(x_1, x_2)\cdot(y_1, y_2) = x_1y_1$$

is not an inner product on $\mathbf{R}^2$. Why?

Solution. Since $(0, x_2)\cdot(0, x_2) = 0$ for all $x = (0, x_2) \in \mathbf{R}^2$, where x_2 is any element in $\mathbf{R}$, in particular, $x_2 \neq 0$, it follows that the product defined above does not satisfy property (ii) in 8.3.

Exercise (8.F.). If $x = (x_1, x_2, \ldots, x_p) \in \mathbf{R}^p$, define $\|x\|_1$ by

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_p|.$$

Prove that $x \mapsto \|x\|_1$is a norm in $\mathbf{R}^p$.

Solution. (i) Since $|x_i| \geq 0$ for all $i \in \{1, 2, \ldots, p\}$, it follows that $\sum_{i=1}^p |x_i| \geq 0$. Thus $\|x\|_1 \geq 0$ for all $x \in \mathbf{R}^p$, showing that the definition above satisfies property (i) in 8.5.

(ii) Since $\sum_{i=1}^p |x_i| = 0$ if and only if $|x_i| = 0$ for all $i \in \{1, 2, \ldots, p\}$, it follows that $x = (0, 0, \ldots, 0)$, showing that the definition above satisfies property (ii) in 8.5.

(iii) For all $a \in \mathbf{R}$ and $x = (x_1, x_2, \ldots, x_p) \in \mathbf{R}^p$, we have

$$\begin{aligned}
\|ax\|_1 &= \|a(x_1, x_2, \ldots, x_p)\|_1 \\
&= \|(ax_1, ax_2, \ldots, ax_p)\|_1 \\
&= |ax_1| + |ax_2| + \cdots + |ax_p| \\
&= |a|\,|x_1| + |a|\,|x_2| + \cdots + |a|\,|x_p| \\
&= |a|\,(|x_1| + |x_2| + \cdots + |x_p|) \\
&= |a|\,\|x\|_1\,,
\end{aligned}$$

showing that the definition above satisfies property (iii) in 8.5.

(iv) For all $x = (x_1, x_2, \ldots, x_p)$ and $y = (y_1, y_2, \ldots, y_p)$ in $\mathbf{R}^p$, we have

$$\begin{aligned}
\|x + y\|_1 &= \|(x_1, x_2, \ldots, x_p) + (y_1, y_2, \ldots, y_p)\|_1 \\
&= \|(x_1 + y_1, x_2 + y_2, \ldots, x_p + y_p)\|_1 \\
&= |x_1 + y_1| + |x_2 + y_2| + \cdots + |x_p + y_p|\,.
\end{aligned}$$

Since $|x_i + y_i| \leq |x_i| + |y_i|$ for all $i \in \{1, 2, \ldots, p\}$, it follows that $\sum_{i=1}^{p} |x_i + y_i| \leq \sum_{i=1}^{p}(|x_i| + |y_i|) = \sum_{i=1}^{p} |x_i| + \sum_{i=1}^{p} |y_i| = \|x\|_1 + \|y\|_1$. Thus

$$\|x + y\|_1 \leq \|x\|_1 + \|y\|_1\,,$$

showing that the definition above satisfies property (iv) in 8.5.

Therefore the definition above satisfies the properties in 8.5.

Exercise (8.G.). If $x = (x_1, x_2, \ldots, x_p) \in \mathbf{R}^p$, define $\|x\|_\infty$ by

$$\|x\|_\infty = \sup\{|x_1|, |x_2|, \ldots, |x_p|\}.$$

Prove that $x \mapsto \|x\|_\infty$is a norm in $\mathbf{R}^p$.

Solution. (i) Since $|x_i| \geq 0$ for all $i \in \{1, 2, \ldots, p\}$, it follows that $\sup\{|x_i| : i \in \{1, 2, \ldots, p\}\} \geq 0$. Thus $\|x\|_\infty \geq 0$ for all $x \in \mathbf{R}^p$, showing that the definition above satisfies property (i) in 8.5.

(ii) Since $\sup\{|x_i| : i \in \{1, 2, \ldots, p\}\} = 0$ if and only if $|x_i| \leq 0$ for all $i \in \{1, 2, \ldots, p\}$ and for $\epsilon > 0$, there exists $j \in \{1, 2, \ldots, p\}$ such that $|x_j| > 0 - \epsilon$ if and only if $x_i = 0$ for all $i \in \{1, 2, \ldots, p\}$, it follows that $x = (0, 0, \ldots, 0)$, showing that the definition above satisfies property (ii) in 8.5.

(iii) For all $a \in \mathbf{R}$ and $x = (x_1, x_2, \ldots, x_p) \in \mathbf{R}^p$, we have

$$\begin{aligned}
\|ax\|_\infty &= \|a(x_1, x_2, \ldots, x_p)\|_\infty \\
&= \|(ax_1, ax_2, \ldots, ax_p)\|_\infty \\
&= \sup\{|ax_1|, |ax_2|, \ldots, |ax_p|\} \\
&= \sup\{|a|\,|x_1|, |a|\,|x_2|, \ldots. |a|\,|x_p|\} \\
&= |a| \sup\{|x_1|, |x_2|, \ldots, |x_p|\} \\
&= |a|\,\|x\|_\infty
\end{aligned}$$

showing that the definition above satisfies property (iii) in 8.5.

(iv) For all $x = (x_1, x_2, \ldots, x_p)$ and $y = (y_1, y_2, \ldots, y_p)$ in $\mathbf{R}^p$, we have

$$\begin{aligned}
\|x + y\|_\infty &= \|(x_1, x_2, \ldots, x_p) + (y_1, y_2, \ldots, y_p)\|_\infty \\
&= \|(x_1 + y_1, x_2 + y_2, \ldots, x_p + y_p)\|_1 \\
&= \sup\{|x_1 + y_1|, |x_2 + y_2|, \ldots, |x_p + y_p|\}.
\end{aligned}$$

Since $|x_i + y_i| \le |x_i| + |y_i|$ for all $i \in \{1, 2, \ldots, p\}$, it follows that

$$\begin{aligned}
\sup\{|x_i + y_i| : i \in \{1, 2, \ldots, p\}\} &\le \sup\{|x_i| + |y_i| : i \in \{1, 2, \ldots, p\}\} \\
&\le \sup\{|x_i| : i \in \{1, 2, \ldots, p\}\} + \sup\{|y_i| : i \in \{1, 2, \ldots, p\}\}.
\end{aligned}$$

Thus

$$\|x + y\|_\infty \le \|x\|_\infty + \|y\|_\infty$$

for all $x = (x_1, x_2, \ldots, x_p)$ and $y = (y_1, y_2, \ldots, y_p)$ in $\mathbf{R}^p$, showing that the definition above satisfies property (iv) in 8.5.

Therefore the definition above satisfies the properties in 8.5.

EXERCISE (8.H.). In the set $\mathbf{R}^2$, describe the sets

$$\begin{aligned}
S_1 &= \{x \in \mathbf{R}^2 : \|x\|_1 < 1\}, \\
S_\infty &= \{x \in \mathbf{R}^2 : \|x\|_\infty < 1\}.
\end{aligned}$$

SOLUTION. The set S_1 is the interior of the square with vertices $(0, \pm 1)$, $(\pm 1, 0)$ and S_∞ is the interior of the square with vertices $(1, \pm 1)$, $(-1, \pm 1)$.

EXERCISE (8.P.). If x, y belongs to $\mathbf{R}^p$, then

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

holds if and only if $x \cdot y = 0$. In this case, one says that x and y are **orthogonal** or **perpendicular**.

SOLUTION. Suppose $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. Then

$$\begin{aligned} \|x+y\|^2 &= \|x\|^2 + \|y\|^2 \\ (x+y)\cdot(x+y) &= \|x\|^2 + \|y\|^2 \\ x\cdot x + y\cdot y + 2(x\cdot y) &= \|x\|^2 + \|y\|^2 \\ \|x\|^2 + \|y\|^2 + 2(x\cdot y) &= \|x\|^2 + \|y\|^2 \\ 2(x\cdot y) &= 0, \end{aligned}$$

it follows that $x \cdot y = 0$.

Conversely, suppose $x \cdot y = 0$. Since, by Parallelogram Identity,

$$\begin{aligned} \|x+y\|^2 + \|x-y\|^2 &= 2(\|x\|^2 + \|y\|^2) \\ \|x+y\|^2 &= 2(\|x\|^2 + \|y\|^2) - \|x-y\|^2 \\ \|x+y\|^2 &= 2(\|x\|^2 + \|y\|^2) - (x-y)\cdot(x-y) \\ \|x+y\|^2 &= 2\|x\|^2 + 2\|y\|^2 - (x\cdot x + y\cdot y - 2(x\cdot y)) \\ \|x+y\|^2 &= 2\|x\|^2 + 2\|y\|^2 - \|x\|^2 - \|y\|^2 + 2(x\cdot y) \\ \|x+y\|^2 &= \|x\|^2 + \|y\|^2 + 2(x\cdot y), \end{aligned}$$

it follows that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

As an alternate proof, we have

$$\begin{aligned} & \|x+y\|^2 && = \|x\|^2 + \|y\|^2 \\ \iff & (x+y)\cdot(x+y) && = \|x\|^2 + \|y\|^2 (\text{since } \|x+y\|^2 = (x+y)\cdot(x+y)) \\ \iff & x\cdot x + y\cdot y + 2(x\cdot y) && = \|x\|^2 + \|y\|^2 (\text{since } xy = yx) \\ \iff & 2(x\cdot y) && = 0 \qquad (\text{since } x\cdot x = \|x\|^2 \text{ and } y\cdot y = \|y\|^2, \text{ so cancel both sides}) \\ \iff & x\cdot y && = 0. \end{aligned}$$

EXERCISE (8.Q.). A subset K of $\mathbf{R}^p$ is said to be convex if, whenever x, y belong to K and t is a real number such that $0 \le t \le 1$, then the point

$$tx + (1 - ty)$$

also belongs to K. Interpret this condition geometrically and show that the subsets

$$\begin{aligned} K_1 &= \{x \in \mathbf{R}^2 : |x| \le 1\}, \\ K_2 &= \{(\xi, \eta) \in \mathbf{R}^2 : 0 < \xi < \eta\}, \\ K_3 &= \{(\xi, \eta) \in \mathbf{R}^2 : 0 \le \eta \le 1\}, \end{aligned}$$

are convex. Show that the subset

$$K_4 = \{x \in \mathbf{R}^2 : \|x\| = 1\}$$

is not convex.

SOLUTION. Since, by Triangle Inequality,

$$\begin{aligned} |tx + (1-t)y| &\leq |tx| + |(1-t)y| \\ &= t\,|x| + (1-t)\,|y| \\ &\leq t(1) + (1-t)(1) \\ &= 1, \end{aligned}$$

for all $x, y \in K_1$ and real numbers $t \in [0,1]$, it follows that $tx+(1-t) \in K_1$.

Since, for all $x = (\xi_1, \eta_1)$ and $y = (\xi_2, \eta_2)$ in K_2 and real number $t \in [0,1]$,

$$\begin{aligned} tx + (1-t)y &= t(\xi_1, \eta_1) + (1-t)(\xi_2, \eta_2) \\ &= (t\xi_1, t\eta_1) + ((1-t)\xi_2, (1-t)\eta_2) \\ &= (t\xi_1 + (1-t)\xi_2, t\eta_1 + (1-t)\eta_2), \end{aligned}$$

and $0 < t\xi_1 + (1-t)\xi_2 < t\eta_1 + (1-t)\eta_2$ (for $0 < \xi_1 < \eta_1, 0 < \xi_2 < \eta_2$ and $0 \leq t, (1-t) \leq 0$), it follows that $tx + (1-t) \in K_2$.

Since, for all $x = (\xi_1, \eta_1)$ and $y = (\xi_2, \eta_2)$ in K_3 and real number $t \in [0,1]$,

$$\begin{aligned} tx + (1-t)y &= t(\xi_1, \eta_1) + (1-t)(\xi_2, \eta_2) \\ &= (t\xi_1, t\eta_1) + ((1-t)\xi_2, (1-t)\eta_2) \\ &= (t\xi_1 + (1-t)\xi_2, t\eta_1 + (1-t)\eta_2). \end{aligned}$$

and

$$0 \leq t\eta_1 + (1-t)\eta_2 \leq t(1) + (1-t)(1) = 1,$$

it follows that $tx + (1-t) \in K_3$.

Counterexample. Let $x = (-1, 0) \in K_4$ and $y = (1,0) \in K_4$. For $t = \frac{1}{2}$

$$tx + (1-t)y = \frac{1}{2}(-1,0) + \frac{1}{2}(1,0) = (0,0) \notin K_4.$$

CHAPTER 9

Open and Closed Sets

EXERCISE (9.B.). Justify the assertions made in Example 9.2(c).

SOLUTION. For each point $(x_0, y_0) \in G$, then $x_0^2 + y_0^2 < 1$. Let $r = 1 - \|(x_0, y_0)\| = 1 - \sqrt{x_0^2 + y_0^2}$, then $r > 0$. For every point (x, y) in $\mathbf{R}^2$ satisfying $\|(x, y) - (x_0, y_0)\| < r$, since

$$\begin{aligned} x^2 + y^2 &= \|(x, y) - (0, 0)\| \\ &\leq \|(x, y) - (x_0, y_0)\| + \|(x_0, y_0) - (0, 0)\| \\ &< r + \sqrt{x_0^2 + y_0^2} \\ &\leq 1 - \sqrt{x_0^2 + y_0^2} + \sqrt{x_0^2 + y_0^2} \\ &= 1, \end{aligned}$$

it follows that $(x, y) \in G$, showing that G is open.

For each point $(x_0, y_0) \in H$, then $0 < x_0^2 + y_0^2 < 1$. Let $r = \inf\{\sqrt{x_0^2 + y_0^2}, 1 - \sqrt{x_0^2 + y_0^2}\}$, then $r > 0$. For every point (x, y) in $\mathbf{R}^2$ satisfying $\|(x, y) - (x_0, y_0)\| < r$, since

$$\begin{aligned} x^2 + y^2 &= \|(x, y) - (0, 0)\| \\ &\leq \|(x, y) - (x_0, y_0)\| + \|(x_0, y_0) - (0, 0)\| \\ &< r + \sqrt{x_0^2 + y_0^2} \\ &\leq 1 - \sqrt{x_0^2 + y_0^2} + \sqrt{x_0^2 + y_0^2} \\ &= 1, \end{aligned}$$

and

$$\begin{aligned}
x^2 + y^2 &= \|(x, y) - (0, 0)\| \\
&\geq \|(x, y) - (x_0, y_0)\| - \|(x_0, y_0) - (0, 0)\| \\
&> r - \sqrt{x_0^2 + y_0^2} \\
&\geq \sqrt{x_0^2 + y_0^2} - \sqrt{x_0^2 + y_0^2} \\
&= 0,
\end{aligned}$$

it follows that $(x, y) \in H$, showing that H is open.

If $z = (1, 0) \in F$, then for any $r > 0$, there is a point $y = (1 + r/2, 0) \in \mathscr{C}(F)$ (for $\|(1 + r/2, 0)\| = 1 + r/2 > 1$) but

$$\begin{aligned}
\|y - z\| &= \left\|(1 + \frac{r}{2}, 0) - (1, 0)\right\| \\
&= \left\|\frac{r}{2}, 0\right\| \\
&= \frac{r}{2} \\
&< r.
\end{aligned}$$

Thus F is not open.

EXERCISE (9.D.). What are the interior, boundary, and exterior points in $\mathbf{R}$ of the set $[0, 1)$.

SOLUTION. Consider the set $[0, 1)$ in $\mathbf{R}$.

Every point in the open interval $(0, 1)$ is an interior point of $[0, 1)$ in $\mathbf{R}$ since the neighborhood $(x - \epsilon, x + \epsilon)$ of $x \in (0, 1)$ where $\epsilon = \min\{1 - x, x\}$ is contained in $[0, 1)$.

The points 0 and 1 are boundary points of $[0, 1)$ in $\mathbf{R}$ since every neigborhood $(-\epsilon, \epsilon)$ of 0 for $1 > \epsilon > 0$ contains point $\epsilon/2$ in $[0, 1)$ and point $-\epsilon/2$ in $\mathbf{R}\backslash[0, 1)$, and simmilarly, every neighborhood $(1 - \epsilon, 1 + \epsilon)$ of 1 for $1 > \epsilon > 0$ contains point $1 - (\epsilon/2)$ in $[0, 1)$ and point $1 + (\epsilon/2)$ in $\mathbf{R}\backslash[0, 1)$.

Every point in the set $\mathbf{R}\backslash[0, 1]$ is an exterior point of $[0, 1)$ in $\mathbf{R}$ since for, if $x < 0$, then the neighborhood $(x - \epsilon, x + \epsilon)$ of $x \in (-\infty, 0)$ where $\epsilon = -x$ does not intersect $[0, 1)$, and if $x > 1$, then the neighborhood $(x - \epsilon, x + \epsilon)$ of $x \in (1, +\infty)$ where $\epsilon = x - 1$ does not intersect $[0, 1)$.

EXERCISE (9.G.). Show that a subset of $\mathbf{R}^p$ is open if and only if it is the union of a countable collections of open balls. (Hint: the set of all points in $\mathbf{R}^p$ all of whose coordinates are rational numbers is countable.)

SOLUTION. Since an open ball is open, it follows from 9.3(c) that the union of any countable union of open balls is open.

Conversely, let $G \neq 0$ be an open set in $\mathbf{R}^p$ and let $\{r_n : n \in \mathbf{N}\}$ be an enumeration of all of the points whose coordinates are rational numbers in G. For each $n \in \mathbf{N}$ let m_n be the smallest natural number such that the cell $B_n = \{z \in \mathbf{R}^p : \|z - r_n\| < 1/m_n\}$ is entirely contained in G. It follows that

$$\bigcup_{n \in \mathbf{N}} B_n \subseteq G.$$

Now let x be an arbitrary point in G and let $m \in \mathbf{N}$ be such that $\{z \in \mathbf{R}^p : \|z - x\| < 2/m\} \subseteq G$. It follows from Theorem 6.10 that there exists a point y in $\{z \in \mathbf{R}^p : \|z - x\| < 1/m\}$ whose coordinates are rational numbers; hence $y \in G$ and so $y = r_n$ for some natural number n. We shall show that if x be an arbitrary point in G, then $x \in B_n = \{z \in \mathbf{R}^p : \|z - r_n\| < 1/m_n\}$.

We proceed by contradiction and suppose that x does not belong to $B_n = \{z \in \mathbf{R}^p : \|z - r_n\| < 1/m_n\}$, then we must have $1/m_n < 1/m$ (for if $1/m < 1/m_n$, then $x \in B_n$); but since it is readily seen that for any $z \in \{z \in \mathbf{R}^p : \|z - r_n\| < 1/m\}$, we have

$$\begin{aligned} \|z - x\| \leq & \|z - r_n\| + \|r_n - x\| \\ < & \frac{1}{m} + \frac{1}{m} (\text{since } r_n = y \in \{z \in \mathbf{R}^p : \|z - x\| < 1/m\}) \\ = & \frac{2}{m}, \end{aligned}$$

so that $z \in \{z \in \mathbf{R}^p : \|z - x\| < 2/m\}$. Hence

$$\{z \in \mathbf{R}^p : \|z - r_n\| < 1/m\} \subseteq \{z \in \mathbf{R}^p : \|z - x\| < 2/m\} \subseteq G$$

In view of the definition of m_n, we obtain $1/m \leq 1/m_n$. This contradicts the fact that $1/m_n < 1/m$. Therefore we have $x \in B_n$ for this value of n. Since $x \in G$ is arbitrary, we infer that

$$G \subseteq \bigcup_{n \in \mathbf{N}} B_n.$$

Therefore G is equal to this union.

EXERCISE (9.H.). Every open subset of $\mathbf{R}^p$ is the union of a countable collection of closed sets.

EXERCISE (9.I.). Every closed subset of $\mathbf{R}^p$ is the intersection of a countable collection of open sets.

SOLUTION. Suppose F is a closed subset of $\mathbf{R}^p$. Then the complement of F, $\mathscr{C}(F)$, is open. By Problem 9.H, $\mathscr{C}(H)$ is the union of a countable collection of closed set, that is,

$$\mathscr{C}(F) = \bigcup_{i \in I} F_i$$

where the index set I is a subset of $\mathbf{N}$ and F_i is a closed set for all $i \in I$. Thus

$$\begin{aligned} F &= \mathscr{C}(C(F)) \\ &= \mathscr{C}\left(\bigcup_{i \in I} F_i\right) \\ &= \bigcap_{i \in I} (\mathscr{C}(F_i)) . \end{aligned}$$

Since F_i is open, so $C(F_i)$ is closed. Thus, since F is the intersection of a countable collection of closed sets, so it is closed.

EXERCISE (9.J.). If A is any subset of $\mathbf{R}^p$, let A^0 denote the union of all open sets which are contained in A; the set A^0 is called the **interior** of A. Note that A^0 is an open set; prove that it is the largest open set contained in A. Prove that

$$A^0 \subseteq A, \qquad (A^0)^0 = A^0$$

$$(A \cap B)^0 = A^0 \cap B^0, \qquad (\mathbf{R}^p) = \mathbf{R}^p.$$

Give an example to show that $(A \cup B)^0 = A^0 \cup B^0$ may not hold.

SOLUTION. The set A^0 is the union of the collection of all open sets in A. Hence any open set $G \subseteq A$ must be contained in A^0. By its definition we must have $A^0 \subseteq A$. It follows that $(A^0)^0 \subseteq A^0$. Since A^0 is open and $(A^0)^0$ is the union of all open sets in A^0, we must have $A^0 \subseteq (A^0)^0$. Therefore $(A^0)^0 = A^0$. Since $A^0 \subseteq A$ and $B^0 \subseteq B$ it follows that $A^0 \cap B^0 \subseteq A \cap B$; but since $A^0 \cap B^0$ is open this implies that $(A^0 \cap B^0) \subseteq (A \cap B)^0$. On the other hand $(A \cap B)^0$ is an open set and is contained in A and B; therefore $(A \cap B)^0 \subseteq A^0$ and $(A \cap B)^0 \subseteq B^0$, whence $(A \cap B)^0 \subseteq A^0 \cap B^0$. Consequently $A^0 \cap B^0 = (A \cap B)^0$. Since $\mathbf{R}^p$ is open, $(\mathbf{R}^p)^0 = \mathbf{R}^p$. Let A

be the set of all rational numbers in $(0, 1)$ and B be the set of all irrational numbers in $(0, 1)$. Then $A^0 \cup B^0 = \emptyset$, while $(A \cup B)^0 = (0, 1)$.

Exercise (9.K.). Prove that a point belongs to A^0 if and only if it is an interior point of A.

Solution. Suppose that $x \in A^0$. Then there exists an open set $G \subseteq A$ such that $x \in G$. In view of Definition 9.7, G is a neighborhood of x which is entirely contained in A. Thus x is an interior point of A.

There is a real number $r > 0$ such that every point y in $\mathbf{R}^p$ satisfying $\|y - x\| < r$ also belongs to the set G. Hence the neighborhood $\{y \in \mathbf{R}^p : \|y - x\| < r\}$ of x is entirely contained in A. By Definition 9.7, x is an interior point of A.

Conversely, suppose that x is an interior point of A. In view of Definition 9.7, there is a neighborhood U of x which is entirely contained in A. Hence there is an open set $G \subseteq U$ containing x. By the definition of A^0, G is contained in A^0so that $x \in A^0$.

Exercise (9.L.). If A is any subset of $\mathbf{R}^p$, let A^- denote the intersection of all closed sets containing A; the set A^- is called the **closure** of A. Note that A^- is an closed set; prove that it is the smallest closed set containing A. Prove that

$$A^- \subseteq A, \qquad (A^-)^- = A^-$$
$$(A \cup B)^- = A^- \cup B^-, \qquad \emptyset^- = \emptyset.$$

Give an example to show that $(A \cap B)^- = A^- \cap B^-$ may not hold.

Solution. The set A^- is the intersection of the collection of all closed sets containing A. Hence any closed set $G \supseteq A$ must contains A^-. By its definition we must have $A^- \supseteq A$. It follows that $(A^-)^- \supseteq A^-$. Since A^- is closed and $(A^-)^-$ is the intersection of all closed sets containing A^-, we must have $A^- \supseteq (A^-)^-$. Therefore $(A^-)^- = A^-$. Since $A^- \supseteq A$ or $B^- \supseteq B$ it follows that $A^- \cup B^- \supseteq A \cup B$; but since $A^- \cup B^-$ is closed this implies that $(A^- \cup B^-) \supseteq (A \cup B)^-$. On the other hand $(A \cup B)^-$ is a closed set and contains A or B; therefore $(A \cup B)^- \supseteq A^-$ or $(A \cup B)^- \supseteq B^-$, whence $(A \cup B)^- \subseteq A^- \cup B^-$. Consequently $A^- \cup B^- = (A \cup B)^-$. Since $\emptyset$ is closed, $\emptyset^- = \emptyset$.

Let A be the set of all rational numbers in $[0, 1]$ and B be the set of all irrational numbers in $(0, 1)$. Then $A^- \cap B^- = [0, 1]$, while $(A \cap B)^- = \emptyset$.

Exercise (9.M.). Prove that a point belongs to A^- if and only if it is either an interior or a boundary point of A.

SOLUTION. First we show that if a point $x \in A^-$, then x is either an interior or a boundary point of A. We proceed by contradiction and suppose that x is neither interior nor a boundary point of A. It is readily seen that there exists a number $r > 0$ such that $\{y \in \mathbf{R}^p : \|y - x\| < r\} \cap A = \emptyset$. Thus

$$\mathbf{R}^p \backslash \{y \in \mathbf{R}^p : \|y - x\| < r\}$$

is closed and

$$A \subset \mathbf{R}^p \backslash \{y \in \mathbf{R}^p : \|y - x\| < r\}.$$

This implies that

$$A^- \subset (\mathbf{R}^p \backslash \{y \in \mathbf{R}^p : \|y - x\| < r\})^-$$

so that

$$A^- \subset \mathbf{R}^p \backslash \{y \in \mathbf{R}^p : \|y - x\| < r\}$$

(cf. Exercise 9.L). From this it follows that

$$x \in \mathbf{R}^p \backslash \{y \in \mathbf{R}^p : \|y - x\| < r\}.$$

This contradicts the fact that $x \in \{y \in \mathbf{R}^p : \|y - x\| < r\}$. Therefore, x is either an interior or a boundary point of A.

Next we show that if x is either an interior or a boundary point of A, then $x \in A^-$. In fact, if x is an interior point of A, then $x \in A \subset A^-$. If x is a boundary point of A, then x is a cluster point of any closed set F_α, $\alpha \in \mathbf{N}$, containing A. Since F is closed, it follows that $x \in F_\alpha$, and hence

$$x \in \bigcap_{\alpha \in \mathbf{N}} F_\alpha.$$

That is, x is in the intersection of all closed sets F_α containing A, showing that $x \in A^-$.

CHAPTER 10

The Nested Cells and Bolzano-Weierstrass Theorem

EXERCISE (10.C.). A point x is a cluster point of a set $A \subseteq \mathbf{R}^p$ if and only if every neighborhood of x contains infinitely many points of A.

SOLUTION. Suppose that x is a cluster point of a set $A \subseteq \mathbf{R}^p$, and we claim that every neighborhood of x contains infinitely many points of A. We proceed by contradiciton and assume that there exists a neighborhood of x containing only a finite number of points of A distinct from x. Let $z_1, z_2, \ldots, z_n$ be such points and pick $\epsilon = \min\{\|x - z_i\| : i \in \{1, 2, \ldots n\}\}$. Hence the neighborhood of x with radius $\epsilon/2$ does not have any $y \in A$; otherwise $y \in \{z_1, z_2, \ldots, z_n\}$ and $\|x - y\| > \epsilon/2$. This contrary to the hypothesis that x is a cluster point of A. Hence very neighborhood of x must contains infinitely many points of A.

Conversely, suppose that every neighborhood of x contains infinitely many points of A. Hence every neighborhood of x contains at least one points of A distinct from x. Then by Defintion 10.3, x is cluster point of A.

EXERCISE (10.D.). Let $A = \{1/n \colon n \in \mathbf{N}\}$. Show that every point of A is a boundary point in $\mathbf{R}$, but that 0 is the only cluster point of A in $\mathbf{R}$.

SOLUTION. Let $\frac{1}{n}$ be an arbitrary number in A and let $\epsilon > 0$ be arbitrary. Then $(\frac{1}{n} - \epsilon, \frac{1}{n} + \epsilon) \subseteq \mathbf{R}$. It follows from Theorem 6.10 that there exists an irrational number y in $(\frac{1}{n} - \epsilon, \frac{1}{n} + \epsilon)$; hence $(\frac{1}{n} - \epsilon, \frac{1}{n} + \epsilon)$ contains a point of A, namely, $\frac{1}{n}$ and a point of $\mathscr{C}(A)$, namely, y. Therefore, from the arbitrary nature of integer n and $\epsilon > 0$, every point of A is a boundary point in $\mathbf{R}$.

Let $\epsilon > 0$ be arbitrary. Then $(-\epsilon, \epsilon) \subseteq \mathbf{R}$. Since $(-\epsilon, \epsilon) \cap A = \{\frac{1}{n} \colon n > \frac{1}{\epsilon}\} \neq 0$, it follows that every neighborhood of 0 contains at least one points of A distinct from 0. Hence 0 is a cluster point of A. We claim that 0 is the only cluster point of A. To see this, let $x \neq 0$ be an arbitrary number

of $\mathbf{R}$. Consider two cases: $x < 0$ and $x > 0$. We will show in each case that x is not a cluster.

If $x < 0$ and ϵ be such that $\epsilon = \frac{|x|}{2}$, then $(x - \frac{|x|}{2}, x + \frac{|x|}{2}) \cap A = \emptyset$; otherwise there exists $n \in \mathbf{N}$ such that $\frac{1}{n} < x + \frac{|x|}{2} < 0$, a contradiction. Hence $x < 0$ is not a cluster point of A.

If $x > 0$ and ϵ be such that $\epsilon = \frac{x - \frac{1}{N}}{2}$ for some $\frac{1}{N} < x$, then $(x - \frac{x-\frac{1}{N}}{2}, x + \frac{x+\frac{1}{N}}{2}) \cap A = \{\frac{1}{n} \colon 0 < n < N\}$. Since the set $\{\frac{1}{n} \colon 0 < n < N\}$ is finite, it follows that the neighborhood $(x - \frac{x-\frac{1}{N}}{2}, x + \frac{x+\frac{1}{N}}{2})$ of x contains a finite number of points of A. Hence $x > 0$ is not a cluster point of A as it would contradict the fact that every neighborhood of x contains infinitely many points of A provided x is a cluster point of A in $\mathbf{R}$ (cf. Excercise 10.C).

Exercise (10.E.). Let A, B be subsets of $\mathbf{R}^p$ and let x be a cluster point of $A \cap B$ in $\mathbf{R}^p$. Prove that x is cluster point of both A and B.

Solution. Suppose that x is a cluster point of $A \cap B$ in $\mathbf{R}^p$. Then every neighborhood U of x contains at least one point y of $A \cap B$ distinct from x, that is, $y \in U \cap (A \cap B) = (U \cap A) \cap (U \cap B)$. Hence $y \in U \cap A$ and $y \in U \cap B$. Therefore x is a cluster point of A since every neighborhood U of x contains at least one point y of A distinct from x and, and x is a cluster point of B since every neighborhood U of x contains at least one point y of A distinct from x.

Exercise (10.F.). Let A, B be subsets of $\mathbf{R}^p$ and let x be a cluster point of $A \cup B$ in $\mathbf{R}^p$. Prove that x is either a cluster point of A or of B.

Solution. Suppose that x is a cluster point of $A \cup B$ in $\mathbf{R}^p$. Then every neighborhood U of x contains at least one point y of $A \cup B$ distinct from x, that is, $y \in U \cap (A \cup B) = (U \cap A) \cup (U \cap B)$. Hence $y \in U \cap A$ or $y \in U \cap B$. If $y \in U \cap A$, then x is a cluster point of A since every neighborhood U of x contains at least one point y of A distinct from x. If $y \in U \cap B$, then x is a cluster point of B since every neighborhood U of x contains at least one point y of B distinct from x.

Exercise (10.G.). Show that every point in the Cantor set $\mathbf{F}$ is a cluster point of both $\mathbf{F}$ and $\mathscr{C}(\mathbf{F})$.

Solution. We observe that the Cantor set F does not contain any non-void interval. For if x belongs to $\mathbf{F}$ and (a, b) is an open interval containing x, then (a, b) contains some middle thirds that were removed to obtain $\mathbf{F}$.

Hence (a, b) is not a subset of the Cantor set, but contains infinitely many points in its complement $\mathscr{C}(\mathbf{F})$. Thus every neighborhood of any point x in F contains points of F distinct from x and intersects its complement $\mathscr{C}(\mathbf{F})$.

CHAPTER 11

The Heinel-Borel Theorem

EXERCISE (11.A.). Show directly from the definition (i.e., without using the Heine-Borel Theorem) that the open ball given by $\{(x, y) : x^2 + y^2 < 1\}$ is not compact in $\mathbf{R}^2$.

SOLUTION. In $\mathbf{R}^2$ we consider the open ball $B = \{(x, y) : x^2 + y^2 < 1\}$. Let $G_n = \{(x, y) : x^2 + y^2 < 1 - \frac{1}{n}\}$ for $n > 1$, so that $\mathscr{G} = \{G_n : n > 1\}$ is a collection of open sets of $\mathbf{R}^2$ whose union contains B. If $\{G_{n_1}, G_{n_2}, \dots, G_{n_k}\}$ is a finite subcollection of $\mathscr{G}$, let $M = \sup\{n_1, n_2, \dots, n_k\}$so that $G_{n_j} \subseteq G_M$ for $j = 1, 2, \dots, k$. It follows that $G_M = \{(x, y) : x^2 + y^2 < 1 - \frac{1}{M}\}$ is the union of $\{G_{n_1}, G_{n_2}, \dots, G_{n_k}\}$. However, the open ball $B = \{(x, y) : x^2 + y^2 < 1\}$ does not belong to G_M and hence does not belong to

$$\bigcup_{j=1}^{k} G_{n_j}.$$

Therefore, no finite union of the sets $\mathscr{G}$ can contain B, and B is not compact.

EXERCISE (11.B.). Show directly that the entire space $\mathbf{R}^2$ is not compact.

SOLUTION. Consider the entire space $\mathbf{R}^2$. If $G_n = \{(x, y) : \|(x, y)\| < n\}$, $n \in \mathbf{N}$, then the collection $\mathscr{G} = \{G_n : n \in \mathbf{N}\}$ of open sets is a covering of $\mathbf{R}^2$. If $\{G_{n_1}, \dots, G_{n_k}\}$ is a finite subcollection of $\mathscr{G}$, let $M = \sup\{n_1, \dots, n_k\}$so that $G_{n_j} \subseteq G_M$ for $j = 1, 2, \dots, k$. It follows that $G_M = \{(x, y) : \|(x, y)\| < M\}$ is the union of the sets $\{G_{n_1}, \dots, G_{n_k}\}$. However, the point (M, M) belongs to $\mathbf{R}^2$but does not belong to G_M. Therefore, no finite subcollection of $\mathscr{G}$ can form a covering of the entire space $\mathbf{R}^2$, so that the entire space $\mathbf{R}^2$ is not compact.

EXERCISE (11.C.). Prove directly that if K is compact in $\mathbf{R}^p$ and F is a closed subset of K, then F is compact in $\mathbf{R}^p$.

Solution. Suppose that K is compact in $\mathbf{R}^p$ and F is a closed subset of K. In $\mathbf{R}^p$ we consider the subset F. Let $\{G_\alpha\}$ is a collection of open sets of $\mathbf{R}^p$ whose union contains F. If the open set$\mathscr{C}(F)$ is adjoined to $\{G_\alpha\}$, we obtain a collection Ω of open sets of $\mathbf{R}^p$ whose union contains K. Since K is compact, there is a finite subcollection Φ of Ω whose union contains K, and hence F. If $\mathscr{C}(F)$ is a member of Φ, we may remove it from Φ and still retain a covering of F. We have thus shown that a finite subcollection of $\{G_\alpha\}$ covers F.

Exercise (11.D.). Prove that if K is a compact subset of $\mathbf{R}$, then K is compact when regarded as a subset of $\mathbf{R}^2$.

Solution. We shall show that $K \times \{0\}$ is closed and bounded. First we show that if K is compact in $\mathbf{R}^p$, then $K \times \{0\}$ is closed. Let (x, y) belongs to $\mathscr{C}(K \times \{0\}) = \mathbf{R}^2 \backslash (K \times \{0\})$. Then $x \in \mathscr{C}(K) = \mathbf{R} \backslash K$ or $y \in \mathscr{C}(\{0\}) = \mathbf{R} \backslash \{0\}$. If $x \in \mathscr{C}(K)$, in view of the compactness of K, it follows from the Heine-Borel Theorem that that there exists a real number $r > 0$ such that the neighborhood $\{s \in \mathbf{R} : |s - x| < r\}$ does not intersect K. However, the neighborhood $\{(x, y) \in \mathbf{R}^2 : \|(x, y)\| < r\} \subset \mathscr{C}(K \times \{0\})$ so that $\{(x, y) \in \mathbf{R}^2 : \|(x, y)\| < r\}$ does not intersect $K \times \{0\}$, showing that $\mathscr{C}(K \times \{0\})$ is open. If $y \in \mathscr{C}(\{0\}) = \mathbf{R} \backslash \{0\}$, a similar argument can be given. Therefore, $K \times \{0\}$ is closed in $\mathbf{R}^2$.

Next we show that if K is compact in $\mathbf{R}^p$, then $K \times \{0\}$ is bounded. Since K and $\{0\}$ are compact in $\mathbf{R}$, it follows from the Heine-Borel Theorem that there exists a strictly positive real number λ such that $|k| \leq \lambda$ for all $k \in K$ and it is clear that $0 < \lambda$. Thus $\|(k, 0)\| \leq \sqrt{2}\lambda$ for all $k \in K$. This proves that $K \times \{0\}$ is bounded.

Exercise (11.E.). By modifying the argument in Example 11.2(d), prove that the interval $J = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ is compact in $\mathbf{R}^2$.

Solution. Consider the set $J = [0, 1] \times [0, 1]$; we shall show that J is compact. Let $\mathscr{G} = \{G_\alpha\}$ be a collection of open subsets of $\mathbf{R}^2$ whose union contains J. The point $(x, y) = (0, 0)$ belongs to some open set in the collection $\mathscr{G}$ and so do points $(x, 0)$ satisfying $a \leq x \leq \epsilon$ for some $\epsilon > 0$. Let x^* be the supremum of those numbers x in $[a, b]$ such that the cell $[a, x] \times \{0\}$ is contained in the union of a finite number of sets in $\mathscr{G}$. Since $(x^*, 0)$ belongs to $[a, b] \times \{0\}$, it follows that x^* is an element of some open set in $\mathscr{G}$. Hence for some $\epsilon > 0$, the cell $[x^* - \epsilon, x^* + \epsilon] \times \{0\}$ is contained in a set G_0 in the collection $\mathscr{G}$. But (by the definition of x^*) the

cell $[a, x^* - \epsilon] \times \{0\}$ is contained in the union of a finite number of sets in $\mathscr{G}$. Hence by adding the single set G_0 to the finite number already needed to cover $[a, x^* - \epsilon] \times \{0\}$, we infer that the set $[a, x * +e] \times \{0\}$ is contained in the union of a finite number of sets in $\mathscr{G}$. This gives a contradiction unless $(x^*, 0) = (b, 0)$.

Exercise (11.G.). Prove the Cantor Intersection Theorem by selecting a point x_n from F_n and then applying the Bolzano-Weierstrass Theorem 10.6 to the set $\{x_n : n \in \mathbf{N}\}$.

Solution. Let F_1 be a non-empty closed, bounded subset of $\mathbf{R}^p$ and let

$$F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$$

be a sequence of non-empty closed sets. It is no loss of generality to assume that each set in the sequence is a "proper" subset of its predecessor. Then there is a point x_n such that

$$x_n \in C_n \backslash C_{n+1}$$

for each integer $n \in \mathbf{N}$. Thus the set $\{x_n : n \in \mathbf{N}\}$ contains infinitely many elements of F_1 and hence $\{x_n : n \in \mathbf{N}\} \subset F_1$. Since $\{x_n : n \in \mathbf{N}\}$ is bounded as F_1 is bounded, it follows from the Bolzano-Weierstrass Theorem that $\{x_n : n \in \mathbf{N}\}$ has a cluster point x in $\mathbf{R}^p$. We shall prove that x belongs to all of the sets $\{F_n : n \in \mathbf{N}\}$. We proceed by contradiction and suppose that $x \notin F_{n_0}$ for some $k_0 \in \mathbf{N}$. Then $x \in \mathscr{C}(F_{n_0})$. Since F_{n_0} is closed, $\mathscr{C}(F_{n_0})$ is open and hence there exists a real number $r > 0$ such that the neighborhood $\{y \in \mathbf{R}^p : |y - x| < r\}$ does not intersect F_{n_0} or $\{y \in \mathbf{R}^p : \|y - x\| < r\} \subset \mathscr{C}(F_{n_0})$ for all $k \geq n_0$. Thus $\{y \in \mathbf{R}^p : \|y - x\| < r\} \cap \{x_n : n \in \mathbf{N}\} = \{x_i : 1 \leq i < n_0\}$. Since the set $\{x_i : 1 \leq i < n_0\}$ is finite, it follows that the neighborhood $\{y \in \mathbf{R}^p : \|y - x\| < r\}$ of x contains a finite number of points of $\{x_n : n \in \mathbf{N}\}$. Hence x is not a cluster point of $\{x_n : n \in \mathbf{N}\}$ as it would contradict the fact that every neighborhood of x contains infinitely many points of $\{x_n : n \in \mathbf{N}\}$ provided x is a cluster point of $\{x_n : n \in \mathbf{N}\}$ in $\mathbf{R}^p$ (cf. Excercise 10.C). This contradicts the fact that x is a cluster point of $\{x_n : n \in \mathbf{N}\}$ in $\mathbf{R}^p$. Therefore $x \in F_n$ for all $n \in \mathbf{N}$.

Exercise (11.H.). If F is closed in $\mathbf{R}^p$ and if

$$d(x, F) = \inf\{\|x - z\| : z \in F\} = 0,$$

then x belongs to F.

Solution. We proceed by contraposition and suppose that $x \notin F$. Since F is a non-empty subset of $\mathbf{R}^p$, it follows from the Nearest Point Theorem that there exists at least one point $y \in F$ such that $\|z - x\| \geq \|y - x\|$ for all $z \in F$. Thus

$$d(x, F) = \inf\{\|x - z\| : z \in F\} \geq \|y - x\| > 0.$$

Therefore if

$$d(x, F) = \inf\{\|x - z\| : z \in F\} = 0,$$

then $x \in F$.

Exercise (11.J.). If F is a non-empty closed set in $\mathbf{R}^p$ and if $x \notin F$, is there a *unique* point of F that is nearest to x?

Solution. In general, the answer is no. For, if $F = [-2, -1] \cup [1, 2]$, then F is a non-empty closed set in $\mathbf{R}$. If $x = 0 \notin F$, then $d(x, F) = \inf\{\|x - z\| : z \in F\}$ attains its value 1 at $z = -1$ and $z = 1$. Hence $z = -1$ and $z = 1$ are two points that are nearest to x.

CHAPTER 12

Connected Sets

Exercise (12.A.). If A and B are connected subset of $\mathbf{R}^p$, give examples to show that $A \cup B$, $A \cap B$, $A \backslash B$ can be either connected or disconnected.

Solution. For the first part, the sets

$$A = \{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : x_1 < 0\},$$
$$B = \{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : x_1 \geq 0\}$$

are connected, and $A \cup B = \mathbf{R}^p$ is connected. The sets

$$A = \{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : x_1 < 0\},$$
$$B = \{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : x_1 < 0\}$$

are connected, but $A \cup B$ is disconnected since the pair A, B forms a disconnection of $A \cup B$

For the second part, the sets

$$A = \{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : x_1 > 0\},$$
$$B = \{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : x_1 < 1\}$$

are connected, and

$$A \cap B = \{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : 0 < x_1 < 1\}$$

is connected. The sets

$$A = \mathbf{R}^p \backslash \{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : x_1 \leq 0, x_2 \geq 0\},$$
$$B = \mathbf{R}^d \backslash \{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : x_1 \geq 0, x_2 \leq 0\}$$

are connected, and

$$A \cap B = \{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : 0 < x_1 < 1\},$$

but

$$\begin{aligned} A \cap B &= \{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : x_1, x_2 > 0 \text{ or } x_1, x_2 < 0\} \\ &= \{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : x_1, x_2 > 0\} \cup \{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : x_1, x_2 < 0\} \end{aligned}$$

since the pair

$$\{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : x_1, x_2 > 0\},$$
$$\{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : x_1, x_2 < 0\}$$

forms a disconnection of $A \cap B$

For the third part, the sets

$$A = \{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : x_1 > 0\},$$
$$B = \{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : x_1 < 1\}$$

are connected, and

$$A \backslash B = \{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : x_1 \geq 1\}$$

is connected.

For the last part, the sets $A = \mathbf{R}^p$,

$$B = \{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : 0 \leq x_1 \leq 1\}$$

are connected, but

$$\begin{aligned} A \backslash B &= \mathbf{R}^p \backslash \{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : 0 \leq x_1 \leq 1\} \\ &= \{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p :< 0\} \cup \{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : x_1 > 1\} \end{aligned}$$

is disconnected since since the pair $\{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p :< 0\}$, $\{(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p : x_1 > 1\}$ forms a disconnection of $A \backslash B$.

Exercise (12.B.). If $C \subseteq \mathbf{R}^p$ is connected and x is a cluster point of C, then $C \cup \{x\}$ is connected.

Solution. We proceed by contradiction and suppose that $C \cup \{x\}$ is disconnected. Let A, B be a disconnection for $C' = C \cup \{x\}$. Then $A \cap C'$ and $B \cap C'$ are disjoint, non-empty, and have union C'. One of these sets must contain x; suppose it is B. Since B is an open set, it is also contains points of C so $C \cap (B \backslash \{x\}) \neq \emptyset$. But then A, $B \backslash \{x\}$ form a disconnection of C. This contradicts the fact that $C \subseteq \mathbf{R}^p$ is connected. Therefore $C \cup \{x\}$, where x is a cluster point of C, is connected.

Exercise (12.C.). If $C \subseteq \mathbf{R}^p$ is connected, show that its closure C^- (see Exercise 9.L) is also connected.

Solution. We proceed by contradiction and suppose that C^- is disconnected. Let A, B be a disconnection for C^-. Then $A \cap C^-$ and $B \cap C^-$ are disjoint, non-empty, and have union C^-. We shall show that $A \cap C$,

$B \cap C$ form a disconnection of C. Indeed, $A \cap C$ and $B \cap C$ are disjoint for $(A \cap C) \subseteq (A \cap C^-)$ and $(B \cap C) \subseteq (B \cap C^-)$, as $C \subseteq C^-$.

We next show that $A \cap C$ and $B \cap C$ are non-empty. Suppose that $A \cap C = \emptyset$. Then $C \subseteq \mathbf{R}^p \backslash A$. Since $\mathbf{R}^p \backslash A$ is closed, it follows that $C^- \subseteq \mathbf{R}^p \backslash A$, by the definition of C^-. Thus $A \cap C^- \neq \emptyset$, a contradiction, and hence $A \cap C = \emptyset$. A similar argument can be given for $B \cap C$.

Finally, we have

$$\begin{aligned}(A \cap C) \cup (B \cap C) &= (A \cup B) \cap C \\ &= (A \cup B) \cap (C^- \cap C) \\ &= \big[(A \cup B) \cap C^-\big] \cap C \\ &= \big[(A \cap C^-) \cup (B \cap C^-)\big] \cap C \\ &= C.\end{aligned}$$

But then $A \cap C$, $B \cap C$ form a disconnection of C. This contradicts the fact that $C \subseteq \mathbf{R}^p$ is connected. Therefore C^- is connected.

Exercise (12.E.). If $K \subseteq \mathbf{R}^p$ is convex (see Exercise 8.Q.), then K is connected.

Solution. If not, then there exist two disjoint non-empty open sets A, B whose union is K. Let $x \in A$ and $y \in B$ and consider the line segment S joining x and y; namely,

$$S = \{x + t(y - x) : t \in \mathbf{I}\}.$$

Let $A_1 = \{t \in R : x + t(y - x) \in A\}$ and let $B_1 = \{t \in R : x + t(y - x) \in B\}$. It is easily seen that A_1 and B_1 are disjoint non-empty open subsets of $\mathbf{R}$ and provide a disconnection for $\mathbf{I}$, contradicting Theorem 8.16.

Exercise (12.F.). The Cantor set $\mathbf{F}$ is widly disconnected. Show that if $x, y \in \mathbf{F}$, $x \neq y$, then there is a disconnection A, B of $\mathbf{F}$ such that $x \in A$, $y \in B$.

Solution. Let $x, y \in \mathbf{F}$ be distinct. Then, $x, y \in F_i$ for all $i \in \mathbf{N}$. Now, since x and y are distinct, by the Archimedean property, there must be $N \in \mathbf{N}$ such that $|x - y| > \frac{1}{3^N}$. Hence, x and y belong to different intervals of F_N. Let C be closed interval in F_n containing x. It follows that $x \in A = \mathbf{F} \cap C$ and $y \in B = \mathbf{F} \backslash C$. It is easily seen that A and B are disjoint non-empty open subsets of $\mathbf{R}$ and provide a disconnection for $\mathbf{F}$.

Exercise (12.H.). Show that the set

$$A = \{(x, y) \in \mathbf{R}^2 : 0 < y \leq x^2, x \neq 0\} \cup \{(0, 0)\}$$

is connected in $\mathbf{R}^2$. However there does not exist a polygonal curve lying entirely in A joining $(0, 0)$ to other points in the set.

Solution. We proceed by contradiction and suppose that A_1, A_2 are open sets forming a disconnection of A. Let $A_0 = \{(x, y) \in \mathbf{R}^2 : y = x^2\} \subset A$. Then $(A_1 \cap A_0) \cap (A_2 \cap A_0) = \emptyset$ as $A_1 \cap A_2 \neq \emptyset$. This contradicts the fact that the graph of $y = x^2$ is a continuous curve. Hence the hypothesis that A is disconnected leads to a contradiction.

However there does not exist a polygonal curve lying entirely in A joining $(0, 0)$ to other points in the set.

CHAPTER 13

The Complex Number System

Part 3

Convergence

CHAPTER 14

Introduction to Sequences

Exercise (14.A.). Let $b \in \mathbf{R}$; show that $\lim(b/n) = 0$.

Solution. Let $b \in \mathbf{R}$ and (x_n) be the sequence in $\mathbf{R}$ where $x_n = b/n$. We shall show that $\lim(b/n) = 0$. To do this let $\epsilon > 0$; according to Corollary 6.7(b) (of the Archimedean Property) there exists a natural number $K(\epsilon)$ such that $1/K(\epsilon) < \epsilon/|b|$. Then, if $n \geq K(\epsilon)$ we have

$$0 < |x_n| = \left|\frac{b}{n}\right| = |b|\,\frac{1}{n} < |b|\,\frac{1}{K(\epsilon)} < |b|\,\frac{\epsilon}{|b|} = \epsilon,$$

whence it follows that $|x_n - 0| < \epsilon$ for $n \geq K(\epsilon)$. Since $\epsilon > 0$ is arbitrary this prove that $\lim(b/n) = 0$.

Exercise (14.B.). Show that $\lim(1/n - 1/(n+1)) = 0$.

Solution. Consider the sequence $X = (1/n - 1/(n+1))$ in $\mathbf{R}$. We shall show that $\lim X = 0$. First we note that

$$0 \leq \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \leq \frac{1}{n}.$$

We want the dominant term to be smaller than a given $\epsilon > 0$ when n is sufficiently large. By Corollary 6.7(b), there exists a natural number $K(\epsilon)$ such that $1/K(\epsilon) < \epsilon$. Then if $n \geq K(\epsilon)$ we have

$$0 \leq \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \leq \frac{1}{n} \leq \frac{1}{K(\epsilon)} < \epsilon,$$

whence it follows that $\left|\left(\frac{1}{n} - \frac{1}{n+1}\right) - 0\right| < \epsilon$ for $n \geq K(\epsilon)$. Since ϵ is arbitrary this show that $\lim X = 0$.

Exercise (14.D.). Let $X = (x_n)$ be a sequence in $\mathbf{R}^p$ which is convergent to x. Show that $\lim(\|x_n\|) = \|x\|$. (Hint: use the Triangle Inequality.)

SOLUTION. Let $\epsilon > 0$. Since $\lim(x_n) = x$, there exists a natural number $K(\epsilon)$ such that if $n \geq K(\epsilon)$ then

$$\|x_n - x\| < \epsilon$$

By the Triangle Inequality, we have

$$|\|x_n\| - \|x\|| < \|x_n - x\|$$

It follows that

$$|\|x_n\| - \|x\|| < \epsilon$$

for all $n \geq K(\epsilon)$. Since $\epsilon > 0$ is arbitrary, we infer that $\lim(\|x_n\|) = \|x\|$.

EXERCISE (14.G.). Let $d \in \mathbf{R}$ satisfy $d > 1$. Use Bernoulli's Inequality to show that the sequence (d^n) is not bounded in $\mathbf{R}$. Hence it is not convergent.

SOLUTION. Let $d \in \mathbf{R}$ satisfy $d > 1$ and consider the sequence (d^n). We shall show that the sequence (d^n) is not bounded. First suppose that $d > 1$. Then $d = 1 + c$ with $c_n > 0$ and hence by Bernoulli's Inequality

$$d^n = (1 + c)^n \geq 1 + nc.$$

Since $1 + nc$ is not bounded, we have d^n is not bounded.

EXERCISE (14.H.). Let $b \in \mathbf{R}$ satisfy $0 < b < 1$; show that $\lim(nb^n) = 0$. (Hint: use the Binomial Theorem as in Example 14.8(e).)

SOLUTION. Let $b \in \mathbf{R}$ satisfy $0 < b < 1$ and consider the sequence $X = (nb^n)$; we shall show that $\lim X = 0$, a rather non-obvious fact. Write $b = 1/(1 + k)$ with $k > 0$. By the Binomial Theorem, when $n > 1$ we have

$$(1 + k)^n = 1 + nk + \frac{n(n-1)}{2}k^2 + \cdots > \frac{n(n-1)}{2}k^2.$$

It follows that $1/(1 + k)^n < 2/[n(n - 1)k^2]$, so that

$$0 < nb^n < \frac{2}{(n-1)k^2}.$$

Now let $\epsilon > 0$ be given. Then there exists $K(\epsilon)$ such that if $n \geq K(\epsilon)$, then $2/[(n - 1)k^2] < \epsilon$; whence it follows that $0 \leq nb^n < \epsilon$ and so

$$0 < nb^n - 0 < \epsilon$$

for $n \geq K(\epsilon)$. Since $\epsilon > 0$ is arbitrary, this proves that $\lim(nb^n) = 0$.

EXERCISE (14.I.). Let $X = (x_n)$ be a sequence of strictly positive real numbers such that $\lim(x_{n+1}/x_n) < 1$. Show that for some r with $0 < r < 1$ and some $C > 0$, then we have $0 < x_n < Cr^n$ for all sufficiently large $n \in \mathbf{N}$. Use this to show that $\lim(x_n) = 0$.

SOLUTION. Let (x_n) be a sequence of positive real numbers such that $L = \lim(x_{n+1}/x_n)$ exists. We shall show that if $L < 1$, then (x_n) converges and $\lim(x_n) = 0$. Indeed, since $x_n \geq 0$ for all $n \in \mathbf{N}$, it follows (why?) that $L > 0$ (cf. Exercise 15.A). Let r be a number such that $L < r < 1$, and let $\epsilon = r - L > 0$. There exists a number $K \in \mathbf{N}$ such that if $n \geq K$ then

$$\left|\frac{x_{n+1}}{x_n} - L\right| < \epsilon.$$

It follows from this that if $n \geq K$, then

$$\left|\frac{x_{n+1}}{x_n}\right| < L + \epsilon = L + (r - L) = r.$$

Therefore, if $n \geq K$, we obtain

$$0 < x_{n+1} < x_n r < x_{n-1} r^2 < \cdots < x_K r^{n-K+1}.$$

If we set x_K/r^K, we see that $0 < x_{n+1} < Cr^{n+1}$ for all $n \geq K$. Since $0 < r < 1$, it follows from Example 14.8(c) that $\lim(r^n) = 0$ and therefore $\lim(x_n) = 0$. (why?) (Cf. Exercise 15.B)

EXERCISE (14.J.). Let $X = (x_n)$ be a sequence of strictly positive real numbers such that $\lim(x_{n+1}/x_n) > 1$. Show that X is not a bounded sequence and hence is not convergent.

SOLUTION. Let (x_n) be a sequence of positive real numbers such that $L = \lim(x_{n+1}/x_n)$ exists. We shall show that if $L > 1$, then (x_n) is not a bounded sequence and hence is not convergent. Let r be a number such that $1 < r < L$, and let $\epsilon = L - r > 0$. There exists a number $K \in \mathbf{N}$ such that if $n \geq K$ then

$$\left|\frac{x_{n+1}}{x_n} - L\right| < \epsilon.$$

It follows from this that if $n \geq K$, then

$$\left|\frac{x_{n+1}}{x_n}\right| > L - \epsilon = L - (L - r) = r.$$

Therefore, if $n \geq K$, we obtain

$$x_{n+1} > x_n r > x_{n-1} r^2 > \cdots > x_K r^{n-K+1}.$$

If we set x_K/r^K, we see that $x_{n+1} > Cr^{n+1}$ for all $n \geq K$. Since $r > 1$, it follows that $\lim(r^n) = +\infty$ and therefore X is not a bounded sequence and hence is not convergent.

Exercise (14.K.). Give an example of convergence sequence (x_n) of strictly positive real numbers such that $\lim(x_{n+1}/x_n) = 1$. Give an example of a divergent sequence with this property.

Solution. Let $(1/n)$ be the sequence of of strictly positive real numbers. By Example 14.8(a), $(1/n)$ is a convergence sequence and $x_n \to 0$, but $x_{n+1}/x_n = n/(n+1) = 1 - 1/(n+1) \to 1$.

Let (n) be the sequence of of strictly positive real numbers. Clearly, (n) is a divergence sequence, but $x_{n+1}/x_n = n + 1/n = 1 + 1/n \to 1$.

Exercise (14.L.). Apply the results of Exercises 14.I and 14.J to the following sequences. (Here $0 < a < 1$, $1 < b$, $c > 0$.)

(a)(a^n), (b)(na^n),
(c)(b^n), (d)(b^n/n),
(e)$(c^n/n!)$, (f)$(2^{3n}/3^{2n})$.

Solution. (a) Consider (a^n) which is a sequence of strictly positive real numbers. Since $\lim(a^{n+1}/a^n) = a < 1$, it follows from Exercise 14.I that (a^n) converges and $\lim(a^n) = 0$.

(b) Consider (na^n) which is a sequence of strictly positive real numbers. Since $\lim((n+1)a^{n+1}/na^n) = a < 1$, it follows from Exercise 14.I that (na^n) converges and $\lim(na^n) = 0$.

(c) Consider (b^n) which is a sequence of strictly positive real numbers. Since $\lim(b^{n+1}/b^n) = b > 1$, it follows from Exercise 14.J that (b^n) diverges.

(d) Consider (b^n/n) which is a sequence of strictly positive real numbers. Since

$$\begin{aligned} \lim\left(\frac{\frac{b^{n+1}}{n+1}}{\frac{b^n}{n}}\right) &= \lim\left(b - \frac{b}{n+1}\right) \\ &= b > 1, \end{aligned}$$

it follows from Exercise 14.J that (b^n/n) diverges.

(e) Consider $(c^n/n!)$ which is a sequence of strictly positive real numbers. Since

$$\begin{aligned} \lim\left(\frac{\frac{c^{n+1}}{(n+1)!}}{\frac{c^n}{n!}}\right) &= \lim\left(\frac{cn!}{(n+1)!}\right) \\ &= \lim\left(\frac{c}{n+1}\right) \\ &= 0 < 1, \end{aligned}$$

it follows from Exercise 14.I that $(c^n/n!)$ converges and $\lim(c^n/n!) = 0$.

(f) Consider $(2^{3n}/3^{2n})$ which is a sequence of strictly positive real numbers. Since

$$\lim\left(\frac{\frac{2^{3(n+1)}}{3^{2(n+1)}}}{\frac{2^{3n}}{3^{2n}}}\right) = \frac{8}{9} < 1$$

it follows from Exercise 14.I that $(2^{3n}/3^{2n})$ converges and $\lim(2^{3n}/3^{2n}) = 0$.

CHAPTER 15

Subsequences and Combinations

EXERCISE (15.A.). If (x_n) and (y_n) are convergent sequences of real numbers and if $x_n \le y_n$ for all $n \in \mathbf{N}$, then $\lim(x_n) \le \lim(y_n)$.

SOLUTION. We first give a special case in which one convergent sequence is a squence of 0 numbers: *if $X = (x_n)$ is a convergent sequence of real numbers and if $x_n \ge 0$ for all $n \in \mathbf{N}$, then* $x = \lim(x_n) \ge 0$. In order to do so, suppose the conclusion is not true and that $x < 0$; then $\epsilon = -x$ is positive. Since X converges to x, there is a natural number K such that

$$x - \epsilon < x_n < x + \epsilon$$

for all $n \ge K$. In particular, we have $x_K < x + \epsilon = x + (-x) = 0$. But this contradicts the hypothesis that $x_n \ge 0$ for all $n \in \mathbf{N}$. Therefore, this contradiction implies that $x \ge 0$.

We now give a useful result that is formally stronger than the result above: *if $X = (x_n)$ and $Y = (y_n)$ are convergent sequences of real numbers and if $x_n \le y_n$ for all $n \in \mathbf{N}$, then* $\lim(x_n) \le \lim(y_n)$. In order to do so, let $z_n = y_n - x_n$ so that $Z = (z_n) = Y - X$ and $z_n \ge 0$ for all $n \in \mathbf{N}$. It follows from Theorem 15.6 and the result above that

$$0 \le \lim Z = \lim(y_n) - \lim(x_n),$$

so that $\lim(x_n) \le \lim(y_n)$.

EXERCISE (15.B.). If $X = (x_n)$ and $Y = (y_n)$ are sequences of real numbers which both converge to c and if $Z = (z_n)$ is a sequence such that $x_n \le z_n \le y_n$ for $n \in \mathbf{N}$, then Z also converges to c.

SOLUTION. Let $c = \lim(x_n) = \lim(y_n)$. If $\epsilon > 0$ is given, then it follows from the convergence of X and Y to c that there exists a natural number K such that if $n \ge K$ then

$$|x_n - c| < \epsilon \qquad \text{and} \qquad |y_n - c| < \epsilon.$$

Since the hypothesis implies that

$$x_n - c \le z_n - c \le y_n - c \qquad\qquad \text{for all } n \in \mathbf{N},$$

it follows (why?) that

$$-\epsilon < z_n - c < \epsilon$$

for all $n \in \mathbf{N}$. Since $\epsilon > 0$ is arbitrary, this implies that $\lim(z_n) = c$.

Exercise (15.C.). For x_n given by the following formulas, either establish the convergence or the divergence of the sequence $X = (x_n)$:

$$(a) x_n = \frac{n}{n+1}, \qquad\qquad (b) x_n = \frac{(-1)^n n}{n+1},$$
$$(c) x_n = \frac{2n}{3n^2+1}, \qquad\qquad (d) x_n = \frac{2n^2+3}{3n^2+1},$$
$$(e) x_n = n^2 - n, \qquad\qquad (f) \sin n.$$

Solution. (a) Let (x_n) be the sequence in $\mathbf{R}$ where $x_n = n/(n+1)$. We shall show that $\lim(n/(n+1)) = 1$. Given $\epsilon > 0$, we want to obtain the inequality

$$\left|\frac{n}{n+1} - 1\right| < \epsilon \tag{15.C.1}$$

when n is n is sufficiently large. We first simplify the expression on the left:

$$\left|\frac{n}{n+1} - 1\right| = \left|1 - \frac{1}{n+1} - 1\right| = \left|\frac{1}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n}.$$

Now if the inequality $1/n < \epsilon$ is satisfied, then the inequality (15.C.1) holds. Thus if $1/K < \epsilon$, then for any $n \ge K$, we also have $1/n < \epsilon$ and hence (15.C.1) holds. Therefore the limit of the sequence is 1.

(b) Let (x_n) be the sequence in $\mathbf{R}$ where $x_n = (-1)^n n/(n+1)$. We shall show that the sequence (x_n) is divergent. The subsequence $(x_{2n}) = (-1)^{2n} 2n/(2n+1)$ converges to 1, and the subsequence $(x_{2n+1}) = (-1)^{2n+1}(2n+1)/(2n+3)$ converges to -1. Therefore, we conclude from Lemma 15.2 that (x_n) is divergent.

(c) Let (x_n) be the sequence in $\mathbf{R}$ where $x_n = 2n/(3n^2+1)$. We shall show that $\lim(2n/(3n^2+1)) = 0$. Given $\epsilon > 0$, we want to obtain the inequality

$$(15.C.2) \qquad \left| \frac{2n}{3n^2+1} - 0 \right| < \epsilon$$

when n is n is sufficiently large. We first simplify the expression on the left:

$$\left| \frac{2n}{3n^2+1} - 0 \right| < \left| \frac{2n}{3n^2} \right| = \left| \frac{2}{3n} \right| = \frac{2}{3n}.$$

Now if the inequality $2/3n < \epsilon$ is satisfied, then the inequality (15.C.2) holds. Thus if $2/3K < \epsilon$, then for any $n \geq K$, we also have $2/3n < \epsilon$ and hence (15.C.2) holds. Therefore the limit of the sequence is 0.

(d) Let (x_n) be the sequence in $\mathbf{R}$ where $x_n = (2n^2+3)/(3n^2+1)$. We shall show that $\lim((2n^2+3)/(3n^2+1)) = 2/3$. Given $\epsilon > 0$, we want to obtain the inequality

$$(15.C.3) \qquad \left| \frac{2n^2+3}{3n^2+1} - \frac{2}{3} \right| < \epsilon$$

when n is n is sufficiently large. We first simplify the expression on the left:

$$\left| \frac{2n^2+3}{3n^2+1} - \frac{2}{3} \right| = \left| \frac{6n^2+9-6n^2-2}{9n^2+3} \right| = \left| \frac{7}{9n^2+3} \right| < \frac{7}{9n^2}.$$

Now if the inequality $7/9n^2 < \epsilon$ is satisfied, then the inequality (15.C.3) holds. Thus if $7/9K^2 < \epsilon$, then for any $n \geq K$, we also have $7/9n < \epsilon$ and hence (15.C.3) holds. Therefore the limit of the sequence is 2/3.

(e) Let (x_n) be the sequence in $\mathbf{R}$ where $x_n = n^2 - n$. We shall show that the sequence $(n^2 - n)$ is divergent. Let x be any real number and consider the neighborhood V of x consisting of the open interval $(x-1, x+1)$. According to the Achimedian Property 6.6 there exists a natural number k_0 such that $x + 1 < k_0$; hence, if $n \geq k_0$, it follows that $x_n = x$ does not belong to V. Therefore the subsequence $X' = (k_0, k_0+2, k_0+6, \ldots)$ of X has no points in V, showing that X does not converge to x.

Exercise (15.E.). If X and Y are sequences in $\mathbf{R}^p$ and if $X \cdot Y$ converges, do X and Y converge and have $\lim X \cdot Y = (\lim X) \cdot (\lim Y)$.

Solution. Let $X = (x_n)$ and $Y = (y_n)$ be sequences in $\mathbf{R}$ where $x_n = n$ and $y_n = 1/n$. We have $X \cdot Y$ is the constant sequence $(1, 1, \ldots)$, which evidently converges to 1, but the sequence (n) is divergent while the sequence $(1/n)$ is convergent.

EXERCISE (15.F.). If $X = (x_n)$ is a positive sequence with converges to x, then $(\sqrt{x})$ converges to $\sqrt{x}$. (Hint: $\sqrt{x_n} - \sqrt{x} = (x_n - x)/(\sqrt{x_n} + \sqrt{x})$ when $x \neq 0$.)

SOLUTION. Since $X = (x_n)$ is a positive sequence with converges to x, it follows that $x = \lim(x_n) \geq 0$ so the assertion makes sense. We now consider the two cases: (i) $x = 0$ and (ii) $x > 0$.

Case (i) If $x = 0$, let $\epsilon > 0$ be given. Since $x_n \to 0$ there exists a natural number K such that if $n \geq K$ then

$$0 \leq x_n = x_n - 0 < \epsilon^2.$$

(Why?) Therefore, $0 \leq \sqrt{x_n} \leq \epsilon$ for $n \geq K$. Since $\epsilon > 0$ is arbitrary, this implies that $\sqrt{x_n} \to 0$.

Case (ii) If $x > 0$, then $\sqrt{x} > 0$ and we note that

$$\sqrt{x_n} - \sqrt{x} = \frac{(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})}{\sqrt{x_n} + \sqrt{x}} = \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}.$$

Since $\sqrt{x_n} + \sqrt{x} \geq \sqrt{x} > 0$, it follows that

$$\left|\sqrt{x_n} - \sqrt{x}\right| \leq \left(\frac{1}{\sqrt{x}}\right) |x_n - x| \, .$$

The convergence of $\sqrt{x_n} \to \sqrt{x}$ follows from the fact that $x_n \to x$.

EXERCISE (15.L.). If $0 < a \leq b$ and if $x_n = (a^n + b^n)^{1/n}$, then $\lim(x_n) = b$.

SOLUTION. Since $0 < a \leq b$, it follows (why?) that $0 < a^n \leq b^n$ for all $n \in \mathbf{N}$. Hence the validity of the last inequality implies the validity of $b^n \leq a^n + b^n \leq 2b^n$ for all $n \in \mathbf{N}$. Thus

$$b \leq (a^n + b^n)^{\frac{1}{n}} \leq 2^{\frac{1}{n}} b.$$

(Why?) Therefore, the convergence of $\lim(a^n + b^n)^{1/n} = b$ follows from the fact that $\lim(b) = b$ and $\lim(2^{\frac{1}{n}} b) = \lim(2^{\frac{1}{n}}) \lim(b) = b$ [see Example 14.8(d) and Theorem 15.4].

EXERCISE (15.N.). Let $A \subseteq \mathbf{R}^p$ and $x \in \mathbf{R}^p$. Then x is a boundary point of A if and only if there is a sequence (a_n) of elemenets in A and a sequence (b_n) of elements in $\mathscr{C}(A)$ such that

$$\lim(a_n) = x = \lim(b_n).$$

SOLUTION. Suppose that $x \in \mathbf{R}^p$ is a boundary point of A. For each natural number n, there exists points $a_n \in A$ and $b_n \in \mathscr{C}(A)$ such that $\|a_n - x\| < 1/n$ and $\|b_n - x\| < 1/n$. Consider the sequences (a_n) and (b_n) in $\mathbf{R}^p$ then. We shall show that $\lim(a_n) = x$ and $\lim(b_n) = x$. To do this let $\epsilon > 0$; according to Corollary 6.7(b) (of the Archimedean Property) there exists a natural number $K(\epsilon)$ such that $1/K(\epsilon) < \epsilon$. Then, if $n \geq K(\epsilon)$ we have

$$\|a_n - x\| < \frac{1}{n} \leq \frac{1}{K(\epsilon)} < \epsilon.$$

Since $\epsilon > 0$ is arbitrary this proves that $\lim(a_n) = x$. A similar argument gives $\lim(b_n) = x$.

Conversely, suppose that there is a sequence (a_n) of elements in A and a sequence (b_n) of elements in $\mathscr{C}(A)$ such that $\lim(a_n) = x = \lim(b_n)$. For each neighborhood V of x there is a natural number K_V such that for all $n \geq K_V$, then a_n in A and b_n in $\mathscr{C}(A)$ belong to V. Therefore x is a boundary point of A.

CHAPTER 16

Two Criteria for Convergence

EXERCISE (16.A.). Let $x_1 \in \mathbf{R}$ satisfy $x_1 > 1$ and let $x_{n+1} = 2 - 1/x_n$ for $n \in \mathbf{N}$. Show that the sequence (x_n) is monotone and bounded. What is its limit?

SOLUTION. Let $X = (x_n)$ be the sequence in $\mathbf{R}$ defined inductively by

$$x_1 > 1, \qquad x_{n+1} = 2 - \frac{1}{x_n} \qquad \text{for } n \in \mathbf{N}.$$

We shall show that $\lim X = 1$. Since $x_1 > 1$, it follows that $1/x_1 < 1$. Hence we have $x_2 = 2 - 1/x_1 > 1$. We show, by induction, that $x_n > 1$ for all $n \in \mathbf{N}$. Indeed, this is true for $n = 1, 2$. If $x_n > 1$ holds for some $k \in \mathbf{N}$, then

$$x_{k+1} = 2 - \frac{1}{x_k} > 1,$$

so that $x_{k+1} > 1$. Therefore $x_n > 1$ for all $n \in \mathbf{N}$.

We now show, by induction, that $x_n > x_{n+1}$ for all $n \in \mathbf{N}$. If $n = 1$, then $x_1 - x_2 = x_1 - 2 + \frac{1}{x_1} = (x_1 - 1)^2/x_1 > 0$, and hence the truth of this assertion has been verified for $n = 1$. Now suppose that $x_k > x_{k+1}$ for some k; then $1/x_k < 1/x_{k+1}$, so that $-1/x_k > -1/x_{k+1}$ whence it follows that

$$x_{k+1} = 2 - \frac{1}{x_k} > 2 - \frac{1}{x_{k+1}} = x_{k+2}.$$

Thus $x_k > x_{k+1}$ implies that $x_{k+1} > x_{k+2}$. Therefore $x_n > x_{n+1}$ for all $n \in \mathbf{N}$.

We have shown that the sequence $X = (x_n)$ is decreasing and bounded below by 1. It follows from the Monotone Convergence Theorem that X converges to a limit that is at least 1. In this case it is not so easy to evaluate $x = \lim X$ by calculating $\inf\{x_n : n \in \mathbf{N}\}$. However, once we know that the limit exists, there is another way to evaluate its value. According to

Lemma 15.2, we have $x = \lim(x_n) = \lim(x_{n+1})$. Using Theorem 15.6, the limit y must satisfy the relation

$$x = 2 - \frac{1}{x}.$$

In finding the roots of this last equation, we obtain $x^2 = 2x - 1$, which has root 1, and hence the limit must equal 1.

EXERCISE (16.B.). Let $y_1 = 1$ and $y_{n+1} = (2 + y_n)^{1/2}$ for $n \in \mathbf{N}$. Show that (y_n) is monotone and bounded. What is its limit?

SOLUTION. Let $Y = (y_n)$ be the sequence in $\mathbf{R}$ defined inductively by

$$y_1 = 1, \qquad y_{n+1} = (2 + y_n)^{1/2} \qquad \text{for } n \in \mathbf{N}.$$

We shall show that $\lim Y = 2$. Direct calculation shows that $y_2 = \sqrt{3}$. Hence we have $y_1 < y_2 < 2$. We show, by induction, that $y_n < 2$ for all $n \in \mathbf{N}$. Indeed, this is true for $n = 1, 2$. If $y_k < 2$ holds for some $k \in \mathbf{N}$, then

$$y_{k+1} = (2 + y_k)^{1/2} < (2 + 2)^{1/2} = 2,$$

so that $y_{k+1} < 2$. Therefore $y_n < 2$ for all $n \in \mathbf{N}$.

We now show, by induction, that $y_n < y_{n+1}$ for all $n \in \mathbf{N}$. The truth of this assertion has been verified for $n = 1$. Now suppose that $y_k < y_{k+1}$ for some k; then $2 + y_k < 2 + y_{k+1}$, whence it follows that

$$y_{k+1} = (2 + y_k)^{1/2} < (2 + y_{k+1})^{1/2} = y_{k+2}.$$

Thus $y_k < y_{k+1}$ implies that $y_{k+1} < y_{k+2}$. Therefore $y_n < y_{n+1}$ for all $n \in \mathbf{N}$.

We have shown that the sequence $Y = (y_n)$ is increasing and bounded above by 2. It follows from the Monotone Convergence Theorem that Y converges to a limit that is at most 2. In this case it is not so easy to evaluate $y = \lim Y$ by calculating $\sup\{y_n : n \in \mathbf{N}\}$. However, once we know that the limit exists, there is another way to evaluate its value. According to Lemma 15.2, we have $y = \lim(y_n) = \lim(y_{n+1})$. Using Theorem 15.6, the limit y must satisfy the relation

$$y = (2 + y)^{1/2}.$$

To find the roots of this last equation, we square to obtain $y^2 = 2+y$, which has roots -1, 2. Evidently -1 cannot be the limit (why?) (cf. Exercise 15.A); hence this limit must equal 2.

Exercise (16.E.). Show that every sequence in $\mathbf{R}$ either has a monotone increasing subsequence or a monotone decreasing subsequence.

Solution. Let $X = (x_n)$ is a sequence of real numbers. For the purpose of this proof, we will say that the mth term x_m is a "peak" if $x_m \geq x_n$ for all n such that $n \geq m$. (That is, x_m is never exceeded by any term that follows it in the sequence.) Note that, in a decreasing sequence, every term is a peak, while in an increasing sequence, no term is a peak.

We will consider two cases, depending on whether X has infinitely many, or finitely many, peaks.

Case 1: X has infinitely many peaks. In this case, we list the peaks by increasing subscripts: $x_{m_1}, x_{m_2}, \dots, x_{m_k}, \dots$. Since each term is a peak, we have

$$x_{m_1} \geq x_{m_2} \geq \cdots x_{m_k} \geq \cdots .$$

Therefore, the subsequence (x_{m_k}) of peaks is a decreasing subsequence of X.

Case 2: X has a finite number (possibly zero) of peaks. Let these peaks be listed by increasing subscripts: $x_{m_1}, x_{m_2}, \dots, x_{m_r}$. Let $s_1 = m_r + 1$ be the first index beyond the last peak. Since x_{s_1} is not a peak, there exists $s_2 > s_1$ such that $x_{s_1} < x_{s_2}$. Since x_{s_2} is not a peak, there exists $s_3 > s_2$ such that $x_{s_2} < x_{s_3}$. Continuing in this way, we obtain an increasing subsequence (X_{s_k}) of X.

Exercise (16.G.). Determine the convergence or divergence of the sequence (x_n), where

$$x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \qquad \text{for } n \in \mathbf{N}.$$

Solution. Let (x_n) be the sequence in $\mathbf{R}$ defined by

$$x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \qquad \text{for} \qquad n \in \mathbf{N}.$$

Since

$$x_n \geq \underbrace{\frac{1}{2n} + \cdots + \frac{1}{2n}}_{n \text{ terms}} = n \cdot \frac{1}{2n} = \frac{1}{2}$$

and

$$x_n \leq \underbrace{\frac{1}{n+1} + \cdots + \frac{1}{n+1}}_{n \text{ terms}} = n \cdot \frac{1}{n+1} = \frac{n}{n+1} < 1,$$

it follows that (x_n) is bounded above by 1. Since

$$x_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2},$$

so that

$$\begin{aligned} x_{n+1} - x_n &= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \\ &= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{2}{2n+2} \\ &= \frac{1}{2n+1} - \frac{1}{2n+2} \\ &= \frac{1}{(2n+1)(2n+2)} \\ &> 0, \end{aligned}$$

it follows that (x_n) is increasing. It follows from the Monotone Convergence Theorem that (x_n) converges.

EXERCISE (16.J.). Show directly that the following sequences are not Cauchy sequences:
(a) $((-1)^n)$,
(b) $(n + (-1)^n / n)$,
(c) $\left(n^2\right)$.

SOLUTION. (a) If $X = (x_n)$ is the sequence in $\mathbf{R}$ defined by

$$x_n = (-1)^n \qquad \text{for } n \in \mathbf{N},$$

and if $m > n$, then

$$x_m - x_n = (-1)^m - (-1)^n.$$

In particular, if $m = n + 1$, we have

$$\begin{aligned} |x_{n+1} - x_n| &= \left|(-1)^{n+1} - (-1)^n\right| \\ &= |(-1)^n [(-1) - 1]| \\ &= |(-1)^n (-2)| \\ &= 2. \end{aligned}$$

This shows that X is not a Cauchy sequence, whence we conclude that X is divergent.

(b) If $X = (x_n)$ is the sequence in $\mathbf{R}$ defined by

$$x_n = n + \frac{(-1)^n}{n} \qquad \text{for } n \in \mathbf{N},$$

and if $m > n$, then

$$x_m - x_n = m + \frac{(-1)^m}{m} - n - \frac{(-1)^n}{n}.$$

In particular, if $m = 2n$, then

$$\begin{aligned} |x_{2n} - x_n| &= \left|2n + \frac{(-1)^{2n}}{2n} - n - \frac{(-1)^n}{n}\right| \\ &= \left|n + \frac{1}{2n} - \frac{(-1)^n}{n}\right| \\ &= \left|n + \frac{1 - 2(-1)^n}{2n}\right| \\ &\geq n - 1 \end{aligned}$$

for all $n \in \mathbf{N}$. This shows that X is not a Cauchy sequence, whence we conclude that X is divergent.

(c) If $X = (x_n)$ is the sequence in $\mathbf{R}$ defined by

$$x_n = n^2 \qquad \text{for } n \in \mathbf{N},$$

and if $m > n$, then

$$x_m - x_n = m^2 - n^2.$$

In particular, if $m = n + 1$, then

$$\begin{aligned}|x_{n+1}-x_n| &= \left|(n+1)^2-n^2\right| \\ &= \left|n^2+2n+1-n^2\right| \\ &= |2n+1| \\ &> 1.\end{aligned}$$

This shows that X is not a Cauchy sequence, whence we conclude that X is divergent.

Exercise (16.M.). Establish the convergence and the limits of the following sequences:
(a) $\left((1+1/n)^{n+1}\right)$,
(b) $((1+1/2n)^n)$,
(c) $((1+2/n)^n)$,
(d) $\left((1+1/(n+1))^{3n}\right)$.

Solution. (a) Let $U=(u_n)$ be the sequence of real numbers defined by $u_n=(1+1/n)^{n+1}$ for $n\in\mathbf{N}$. Since

$$\left(1+\frac{1}{n}\right)^{n+1}=\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right),$$

so the sequence (u_n) is the combination of the convergent sequences $((1+1/n)^n)$ and $((1+1/n))$, and hence it follows from Theorem 15.6 that

$$\lim((1+1/n)^{n+1})=\lim((1+1/n)^n)\lim((1+1/n)).$$

The convergence of $((1+1/n)^{n+1})\to e$ follows from the fact that $((1+1/n)^n)\to e$ and $((1+1/n))\to 1$.

(b) Let $U=(u_n)$ be the sequence of real numbers defined by $u_n=(1+1/2n)^n$ for $n\in\mathbf{N}$. Since

$$\begin{aligned}\left(1+\frac{1}{2n}\right)^n &= \left[\left(1+\frac{1}{4n}\right)\left(1+\frac{1}{4n+1}\right)\right]^n \\ &= \left(1+\frac{1}{4n}\right)^n\left(1+\frac{1}{4n+1}\right)^n \\ &= \left[\left(1+\frac{1}{4n}\right)^{4n}\right]^{\frac{1}{4}}\left[\left(1+\frac{1}{4n+1}\right)^{4n+1}\left(\frac{1}{1+\frac{1}{4n+1}}\right)\right]^{\frac{1}{4}},\end{aligned}$$

so the sequence (u_n) is the combination of the convergent sequences $((1+1/4n)^{4n})$, $((1+1/(4n+1))^{4n+1})$ and $(1/(1+1/(4n+1)))$, and hence it follows from Theorem 15.6 that

$$\lim\left(\left(1+\frac{1}{2n}\right)^n\right)=\left[\lim\left(\left(1+\frac{1}{4n}\right)^{4n}\right)\right]^{\frac{1}{4}}\left[\lim\left(\left(1+\frac{1}{4n+1}\right)^{4n+1}\right)\right]^{\frac{1}{4}}\left[\lim\left(\frac{1}{1+\frac{1}{4n+1}}\right)\right]^{\frac{1}{4}}.$$

The convergence of $((1+1/2n)^n)\to\sqrt{e}$ follows from the fact that $((1+1/4n)^{4n})\to e$, $((1+1/(4n+1))^{4n+1})\to e$, and $(1/(1+1/(4n+1)))\to 1$.

(c) Let $U=(u_n)$ be the sequence of real numbers defined by $u_n=(1+2/n)^n$ for $n\in\mathbf{N}$. Since

$$\begin{aligned}\left(1+\frac{2}{n}\right)^n &= \left[\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\right]^n\\ &= \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n+1}\right)^n\\ &= \left(1+\frac{1}{n}\right)^n\left[\left(1+\frac{1}{n+1}\right)^{n+1}\left(\frac{1}{1+\frac{1}{n+1}}\right)\right],\end{aligned}$$

so the sequence (u_n) is the product of the convergent sequences $((1+1/n)^n)$, $((1+1/(n+1))^{n+1})$ and $(1/(1+1/(n+1)))$, and hence it follows from Theorem 15.6 that

$$\lim\left(\left(1+\frac{2}{n}\right)^n\right)=\lim\left(\left(1+\frac{1}{n}\right)^n\right)\lim\left(\left(1+\frac{1}{n+1}\right)^{n+1}\right)\lim\left(\frac{1}{1+\frac{1}{n+1}}\right).$$

The convergence of $((1+2/n)^n)\to e^2$ follows from the fact that $((1+1/n)^n)\to e$, $((1+1/(n+1))^{n+1})\to e$, and $(1/(1+1/(n+1)))\to 1$.

(d) Let $U=(u_n)$ be the sequence of real numbers defined by $u_n=(1+1/(n+1))^{3n}$ for $n\in\mathbf{N}$. Since

$$\begin{aligned}&\left(1+\frac{1}{n+1}\right)^{3n}\\ &=\left(1+\frac{1}{n+1}\right)^n\left(1+\frac{1}{n+1}\right)^n\left(1+\frac{1}{n+1}\right)^n\\ &=\left[\left(1+\frac{1}{n+1}\right)^{n+1}\left(\frac{1}{1+\frac{1}{n+1}}\right)\right]\left[\left(1+\frac{1}{n+1}\right)^{n+1}\left(\frac{1}{1+\frac{1}{n+1}}\right)\right]\left[\left(1+\frac{1}{n+1}\right)^{n+1}\left(\frac{1}{1+\frac{1}{n+1}}\right)\right],\end{aligned}$$

so the sequence (u_n) is the product of the convergent sequences $((1+1/(n+1))^{n+1})$ and $(1/(1+1/(n+1)))$, and hence it follows from Theorem 15.6 that

$$\lim\left(\left(1+\frac{1}{n+1}\right)^{3n}\right) = \left[\lim\left(\left(1+\frac{1}{n+1}\right)^{n+1}\right)\right]^3 \left[\lim\left(\frac{1}{1+\frac{1}{n+1}}\right)\right].$$

The convergence of $\left((1+1/(n+1))^{3n}\right) \to e^3$ follows from the fact that $((1+1/(n+1))^{n+1}) \to e$, and $(1/(1+1/(n+1))) \to 1$.

Exercise (16.N.). Let $0 < a_1 < b_1$ and define, for $n \in \mathbf{N}$,

$$a_{n+1} = (a_n b_n)^{1/2}, \qquad\qquad b_{n+1} = \frac{1}{2}(a_n + b_n).$$

By induction show that $a_n < b_n$. Show that (a_n) and (b_n) converges to the same limit.

Solution. For the first part, we proceed by mathematical induction to show that $a_n < a_{n+1} < b_{n+1} < b_n$. As with all induction arguments, we need a base case and an induction step. We start with the base case $n = 2$. We need to prove that $a_2 < a_3 < b_3 < b_2$. First we demonstrate that $a_2 < b_2$. We know that $(\sqrt{a_1} - \sqrt{b_1})^2 > 0$ since $a_1 < b_1$.

$$(\sqrt{a_1} - \sqrt{b_1})^2 > 0 \implies a_1 - 2\sqrt{a_1}\sqrt{b_1} + b_1 > 0 \implies a_1 + b_1 > 2\sqrt{a_1}\sqrt{b_1} \implies \frac{a_1+b_1}{2} > \surd \tag{16.N.1}$$

Using this fact, we can show:

$$b_2 > a_2 \implies 2b_2 > a_2 + b_2 \implies b_2 > \frac{a_2+b_2}{2} \implies b_2 > b_3.$$

$$b_2 > a_2 \implies a_2 b_2 > a_2^2 \text{ (true, because } a_2 > 0) \implies \sqrt{a_2 b_2} > a_2 \implies a_3 > a_2.$$

Finally, we use a variation on argument (16.N.1) to show that $b_3 > a_3$:

$$(\sqrt{a_2} - \sqrt{b_2})^2 > 0 \implies a_2 - 2\sqrt{a_2}\sqrt{b_2} + b_2 > 0 \implies \frac{a_2+b_2}{2} > \sqrt{a_2 b_2} \implies b_2 > a_2.$$

Putting these together, we find that:

$$b_2 > b_3 > a_3 > a_2.$$

We now proceed to the induction step. This is where we assume that $b_k > b_{k+1} > a_{k+1} > a_k$ for some $k \in \mathbf{N}$, and we need to prove that

$b_{n+1} > b_{n+2} > a_{n+2} > a_{n+1}$. These arguments are going to be similar to the ones in the previous step:

$$b_{k+1} > a_{k+1} \implies 2b_{k+1} > a_{k+1} + b_{k+1} \implies b_{k+1} > \frac{a_{k+1} + b_{k+1}}{2} \implies b_{k+1} > b_{k+2}.$$

$$b_{k+1} > a_{k+1} \implies a_{k+1}b_{k+1} > a_{k+1}^2 \implies \sqrt{a_{k+1}b_{k+1}} > a_{k+1} \implies a_{k+2} > a_{k+1}.$$

$$\begin{aligned} & & (\sqrt{a_{k+1}} - \sqrt{b_{k+1}})^2 > 0 \\ \implies & & a_{k+1} - 2\sqrt{a_{k+1}}\sqrt{b_{k+1}} + b_{k+1} > 0 \\ \implies & & \frac{a_{k+1} + b_{k+1}}{2} > \sqrt{a_{k+1}b_{k+1}} \\ \implies & & b_{k+2} > a_{k+2}. \end{aligned}$$

This gives us the result:

$$b_{k+1} > b_{k+2} > a_{k+2} > a_{k+1}$$

for all $k \geq 3$. Therefore the assertion is true for all $n \in \mathbf{N}$.

For the second part, we note that both (a_n) and (b_n) are bounded above by b_1 and below by a_1. The sequence (a_n) is monotone increasing, and the sequence (b_n) is monotone decreasing. Therefore, by the Monotone Convergence Theorem, both sequences converge.

Let $a = \lim(a_n)$ and $b = \lim(b_n)$. We can take either recurrence relation, take the limit as $n \to \infty$, and we will find that $a = b$. Starting with the recurrence relation for a_n:

$$\lim(a_{n+1}) = \lim(\sqrt{a_n b_n}) \implies a = \sqrt{ab} \implies a^2 = ab \implies a = b.$$

Starting with the recurrence relation for (b_n):

$$\lim(b_{n+1}) = \lim\left(\frac{a_n + b_n}{2}\right) \implies b = \frac{a+b}{2} \implies \frac{b}{2} = \frac{a}{2} \implies a = b.$$

Exercise (16.Q.). Prove that if K_1 and K_2 are compact subsets of $\mathbf{R}^p$, then there exist points $x_1 \in K_1$, $x_2 \in K_2$ such that if $z_1 \in K_1$, $z_2 \in K_2$, then $\|z_1 - z_2\| \geq \|x_1 - x_2\|$.

CHAPTER 17

Sequences of Functions

EXERCISE (17.A.). For each $n \in \mathbf{N}$, let f_n be defined for $x > 0$ by $f_n(x) = 1/(nx)$. For what values of x does $\lim(f_n(x))$ exist?

SOLUTION (1). Let $D = \{x \in \mathbf{R} : x > 0\}$ and for $n \in \mathbf{N}$, let $f_n(x) = 1/(nx)$ and let $f(x) = 0$ for $x \in D$. We have $\lim(1/n) = 0$. Hence it follows from Theorem 15.6 that

$$\lim(f_n(x)) = \lim(1/(nx)) = (1/x)\lim(1/n) = (1/x) \cdot 0 = 0$$

for all $x \in D$.

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SOLUTION (2). Let $D = \{x \in \mathbf{R} : x > 0\}$ and, for each natural number n, let f_n be defined by $f_n(x) = 1/(nx)$. If f defined to be the zero function $f(x) = 0$, $x \in D$, then $f = \lim(f_n)$ on D. Indeed, for any $x \in D$, we have

$$|f_n(x) - f(x)| = \left|\frac{1}{nx}\right| = \frac{1}{nx}$$

If $\epsilon > 0$, there exists a natural number $K(\epsilon, x)$ such that if $n \geq K(\epsilon, x)$, then $1/nx < \epsilon$. Hence for $n \geq K(\epsilon, x)$ we conclude that

$$|f_n(x) - f(x)| < \epsilon.$$

Therefore, we infer that the sequence (f_n) converges to f.

EXERCISE (17.B.). For each $n \in \mathbf{N}$, let g_n be defined for $x \geq 0$ by the formula

$$\begin{aligned} g_n(x) &= nx, 0 \leq x \leq 1/n \\ &= \frac{1}{nx}, 1/n < x. \end{aligned}$$

Show that $\lim(g_n(x)) = 0$ for all $x > 0$.

SOLUTION. Let $D = \{x \in \mathbf{R} : x \geq 0\}$ and, for each natural number n, let g_n be defined by

$$\begin{aligned} g_n(x) &= nx, 0 \leq x \leq 1/n \\ &= \frac{1}{nx}, 1/n < x. \end{aligned}$$

If g defined to be the zero function $g(x) = 0$, $x \in D$, then $g = \lim(g_n)$ for all $x > 0$. Indeed, let $x > 0$; according to Corollary 6.7(b) (of the Archimedean Property) there exists a natural number $K(x)$ such that $1/K(x) < x$. Then, if $n \geq K(x)$ we have

$$\frac{1}{n} \leq \frac{1}{K(x)} < x,$$

whence it follows that $g_n(x) = 1/nx$ for $n \geq K(x)$. We shall show that $\lim(1/(nx)) = 0$. If $\epsilon > 0$ is given, then there exists a natural number $K(x\epsilon)$ such that $1/K(x\epsilon) < x\epsilon$. then if $n \geq K(xe)$ we have

$$\frac{1}{n} \leq \frac{1}{K(x\epsilon)} < x\epsilon,$$

whence it follows that $1/nx < \epsilon$ for $n \geq K(x\epsilon)$. Thus if we choose $K = \max(K(x), K(x\epsilon))$, then we will have

$$|g_n(x) - 0| = \left| \frac{1}{nx} - 0 \right| < \epsilon$$

for all $n \geq K$. Therefore, we conclude that (g_n) converges on D to g.

EXERCISE (17.D.). Show that, if we define f_n on $\mathbf{R}$ by

$$f_n(x) = \frac{nx}{1 + n^2x^2},$$

then (f_n) converges on $\mathbf{R}$.

SOLUTION. Let $D = \mathbf{R}$ and, for each natural number n, let f_n be defined by $f_n(x) = nx/(1 + n^2x^2)$. If f defined to be the zero function $f(x) = 0$, $x \in D$, then $f = \lim(f_n)$ on D. Indeed, if $x = 0$, then $f_n(x) = f_n(0) = 0$ so that $f_n(0) \to 0$. If $x \neq 0$, then we have

$$|f_n(x) - f(x)| = \left| \frac{nx}{1 + n^2x^2} \right| < \frac{|nx|}{n^2x^2} = \frac{1}{n\,|x|}.$$

If $\epsilon > 0$, there exists a natural number $K(\epsilon, x)$ such that if $n \geq K(\epsilon, x)$, then $1/n\,|x| < \epsilon$. Hence for $n \geq K(\epsilon, x)$ we conclude that

$$|f_n(x) - f(x)| < \epsilon.$$

Therefore, we infer that the sequence (f_n) converges to f.

EXERCISE (17.E.). Let h_n be defined on the interval $I = [0, 1]$ by the formula

$$\begin{aligned} h_n(x) =& 1 - nx 0 \leq x \leq 1/n, \\ =& \quad 0, 1/n < x \leq 1. \end{aligned}$$

Show that $\lim(h_n)$ exists on I.

SOLUTION. Let $D = I = [0, 1]$ and, for each natural number n, let h_n be defined by

$$\begin{aligned} h_n(x) =& 1 - nx 0 \leq x \leq 1/n, \\ =& \quad 0, 1/n < x \leq 1. \end{aligned}$$

If h defined by

$$\begin{aligned} h(x) =& 1, x = 0, \\ =& 0, 0 < x \leq 1, \end{aligned}$$

then $f = \lim(h_n)$ for all $x \in I$. Indeed, if $x = 0$, then $h_n(x) = h_n(0) = 1 - n(0) = 1$ so that $h_n(0) \to 1$. If $0 < x \leq 1$; according to Corollary 6.7(b) (of the Archimedean Property) there exists a natural number $K(x)$ such that $1/K(x) < x$. Then, if $n \geq K(x)$ we have

$$\frac{1}{n} \leq \frac{1}{K(x)} < x,$$

whence it follows that $h_n(x) = 1 - nx$ for $n \geq K(x)$. We shall show that $\lim(1 - nx) = 0$. If $\epsilon > 0$ is given, then there exists a natural number $K(\epsilon, x)$ such that $K(\epsilon, x) > (\epsilon + 1)/x$. then if $n \geq K(\epsilon, x)$ we have

$$n > \frac{\epsilon + 1}{x},$$

whence it follows that $1 - nx < \epsilon$ for $n \geq K(\epsilon, x)$. Thus if we choose $K = \max(K(x), K(x\epsilon))$, then we will have

$$|h_n(x) - 0| = |1 - nx - 0| < \epsilon$$

for all $n \geq K$. Therefore, we conclude that (h_n) converges on I to h.

Exercise (17.L.). Show that the convergence in Exercise 17.B is not uniform on the domain $x \geq 0$, but that it is uniform on a set $x \geq c$, where $c > 0$.

Solution. Referring to Exercise 17.B, we consider the functions

$$\begin{aligned} g_n(x) &= nx, 0 \leq x \leq 1/n \\ &= \frac{1}{nx}, 1/n < x. \end{aligned}$$

and the zero function $g(x) = 0$ on $D = \{x \in \mathbf{R} : x \geq 0\}$. If $n_k = k$ and $x_k = 1/k$, then there exists $\epsilon_0 = 1$ satisfying

$$|g_k(x_k) - g(x_k)| = \left| k\frac{1}{k} - 0 \right| = 1.$$

This shows that (g_k) does not converge uniformly on D to f.

We cannot apply Lemma 17.9 as a tool in examining the sequence of function $(g_n(x))$ for uniform convergence, since the functions g_n are not bounded on $\{x \in \mathbf{R} : x \geq 0\}$, which was given as the domain. We choose a smaller domain, taking $E = [c, +\infty)$ with $c > 0$.

Indeed, let $x > c > 0$; according to Corollary 6.7(b) (of the Archimedean Property) there exists a natural number $K(c)$ such that $1/K(c) < c$. Then, if $n \geq K(c)$ we have

$$\frac{1}{n} \leq \frac{1}{K(c)} < c < x,$$

whence it follows that $g_n(x) = 1/nx$ for $n \geq K(c)$. If $\epsilon > 0$ is given, then there exists a natural number $K(c\epsilon)$ such that $1/K(c\epsilon) < c\epsilon$. then if $n \geq K(ce)$ we have

$$\frac{1}{n} < \frac{1}{K(c\epsilon)} < c\epsilon < x\epsilon,$$

whence it follows that $1/nx < \epsilon$ for $n \geq K(c\epsilon)$. Thus if we choose $K = \max(K(c), K(c\epsilon))$ (depending on ϵ but not on $x \in E$), then we will have

$$|g_n(x) - g(x)| = \left| \frac{1}{nx} - 0 \right| = \frac{1}{nx} < \epsilon$$

for all $n \geq K$. Therefore, we conclude that (g_n) converges uniformly on E to g.

Exercise (17.M.). Is the convergence in Exercise 17.D uniform on $\mathbf{R}$?

Solution. Referring to Exercise 17.D, we consider the functions

$$f_n(x) = \frac{nx}{1 + n^2x^2},$$

and the zero function $f(x) = 0$ on $D = \mathbf{R}$. Then $f = \lim(f_n)$ on D. However, the sequence (f_n) does not converges uniformly on $D = \mathbf{R}$ to f. Indeed, if $n_k = k$ and $x_k = \frac{1}{k}$, then

$$|f_k(x_k) - f(x_k)| = \left| \frac{k\frac{1}{k}}{1 + (k)^2 \left(\frac{1}{k}\right)^2} - 0 \right| = \frac{1}{2},$$

showing that (f_n) does not converge uniformly on D to f.

CHAPTER 18

The Limit Superior

Exercise (18.A). Determine the limit superior and the limit inferior of the following bounded sequences in $\mathbf{R}$.

(a) $((-1)^n)$,

(b) $((-1)^n/n)$,

(c) $((-1)^n + 1/n)$,

(d) $(\sin n)$.

Solution. (a) $\limsup((-1)^n) = 1$, $\liminf((-1)^n) = -1$.
(b) $\limsup((-1)^n/n) = \liminf((-1)^n/n) = 0$.
(c) $\limsup((-1)^n + 1/n) = 1$, $\liminf((-1)^n + 1/n) = -1$.

Exercise (18.D.). Give direct proof of Theorem 18.3(c).

Solution. To prove (d), let $v < \liminf(x_n)$ and $u < \limsup(y_n)$; by definition there are only a finite number of natural numbers n such that $v > x_n$ and a finite number such that $u > y_n$. Therefore there can be only a finite number of n such that $v + u > x_n + y_n$, showing that $\liminf(x_n + y_n) \geq \liminf(x_n) + \liminf(y_n)$. This proves statement (c).

Exercise (18.F.). If $X = (x_n)$ is a bounded sequence of strictly positive elements in $\mathbf{R}$ show that $\limsup(x_n^{1/n}) \leq \limsup(x_{n+1}/x_n)$.

Solution. If $\limsup(x_{n+1}/x_n) = +\infty$, then the result is obvious.

If $\limsup(x_{n+1}/x_n) = M < +\infty$, then for each $\epsilon > 0$ there is a natural number $N(\epsilon)$ such that

$$\frac{x_{n+1}}{x_n} \leq M + \epsilon, \qquad n \geq N(\epsilon).$$

Therefore, if $n \geq N(\epsilon)$, we obtain

$$\frac{x_n}{x_{N(\epsilon)}} = \frac{x_{N(\epsilon)+1}}{x_{N(\epsilon)}} \cdots \frac{x_n}{x_{n-1}} \leq (M + \epsilon)^{n-N(\epsilon)}.$$

Hence, if $n \geq N(\epsilon)$

$$x_n \leq \frac{x_{N(\epsilon)}}{(M+\epsilon)^{N(\epsilon)}}(M+\epsilon)^n.$$

This implies

$$x_n^{1/n} \leq \left(\frac{x_{N(\epsilon)}}{(M+\epsilon)^{N(\epsilon)}}\right)^{1/n}(M+\epsilon), \qquad n \geq N(\epsilon).$$

Since

$$\left(\frac{x_{N(\epsilon)}}{(M+\epsilon)^{N(\epsilon)}}\right)^{1/n}(M+\epsilon) \to (M+\epsilon),$$

it follows that $x_n^{1/n} \leq M+\epsilon$ for all $n \geq N(\epsilon)$. Thus there are at most finitely many n such that $x_n^{1/n} > M+\epsilon$, so that $\limsup x_n^{1/n} \leq M+\epsilon$, for all $\epsilon > 0$. Since $\epsilon > 0$ is arbitrary, $\limsup x_n^{1/n} \leq \limsup(x_{n+1}/x_n)$.

Exercise (18.I.). Show that $\limsup X = +\infty$ if and only if there is a subsequence X' of X such that $\lim X' = +\infty$.

Solution. Suppose that $\limsup X = x^* = +\infty$. Let $\epsilon = 1/n > 0$. By Theorem 18.2, there are an infinite number $n \in \mathbf{N}$ such that $x^* - 1/n < x_n$. So there exists a natural number m_n such that $x_{m_n} > x^* - 1/n = +\infty$. Then the subsequence (x_{m_n}) of the sequence (x_n) converges to $+\infty$.

Conversely, suppose that there is a subsequence X' of X such that $\lim X' = +\infty$. Then $v_m = \sup\{x_n : n \geq m\} = +\infty$. But $\limsup(x_n) = \lim(v_m) = +\infty$.

CHAPTER 19

Some Extensions

Part 4

Continuous Functions

CHAPTER 20

Local Properties of Continuous Functions

Exercise (20.A.). Prove that if f is defined for $x \geq 0$ by $f(x) = \sqrt{x}$, then f is continuous at every point of its domain.

Solution. Let $D(f) = [0, +\infty)$ and let f be the "square root" function defined by $f(x) = \sqrt{x}$, $x \in D(f)$. Let a belong to $D(f)$ and let $\epsilon > 0$; then $|f(x) - f(a)| = |\sqrt{x} - \sqrt{a}|$. We wish to make the above expression less than ϵ by making $|x - a|$ sufficiently small. If $a = 0$, then we choose $\delta(\epsilon) = \epsilon^2$. If $a \neq 0$, then $|\sqrt{x} - \sqrt{a}| = |x - a| \, / \, |\sqrt{x} + \sqrt{a}|$ and we want to obtain a bound for $1/ \, |\sqrt{x} + \sqrt{a}|$ on a neighborhood of a. We note that $|\sqrt{x} + \sqrt{a}| \geq |\sqrt{a}|$, so that $1/ \, |\sqrt{x} + \sqrt{a}| \leq 1/ \, |\sqrt{a}|$. Hence

$$\leq \frac{|x - a|}{|\sqrt{a}|}. \tag{20.A.1}$$

Thus if we define $\delta(\epsilon) = \sqrt{a}\epsilon$, then when $|x - a| < \delta(\epsilon)$, the inequality (20.A.1) holds and we have $|f(x) - f(a)| < \epsilon$.

Exercise (20.B.). Show that a "polynomial function"; that is, a function f with the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad \text{for } x \in \mathbf{R},$$

is continuous at every point of $\mathbf{R}$.

Solution. Let $D(f_j) = \mathbf{R}$ and let f_j be a "polynomial function" function defined by $f_j(x) = a_j x^j$, $x \in \mathbf{R}$. Let a belong to $\mathbf{R}$ and let $\epsilon > 0$; then

$$\begin{aligned}
|f_j(x) - f_j(a)| &= \left|a_j x^j - a_j a^j\right| \\
&= \left|a_j(x^j - a^j)\right| \\
&= |a_j| \left|x^j - a^j\right| \\
&= |a_j| \left|(x-a)(x^{j-1} + x^{j-2}a + \cdots + xa^{j-2} + a^{j-1})\right| \\
&= |a_j| \, |x-a| \left|x^{j-1} + x^{j-2}a + \cdots + xa^{j-2} + a^{j-1}\right|.
\end{aligned}$$

We wish to make the above expression less than ϵ by making $|x-a|$ sufficiently small. If $a = 0$, then we choose $\delta(\epsilon) = \sqrt[j]{\epsilon / |a_j|}$. If $a \neq 0$, then we want to obtain a bound for $|a_j| \left|x^{j-1} + x^{j-2}a + \cdots + xa^{j-2} + a^{j-1}\right|$ on a neigborhood of a. For example, if $|x-a| < |a|$, then $0 < |x| < 2|a|$. Hence

$$\begin{aligned}
|f_j(x) - f_j(a)| &= |a_j| \, |x-a| \left|x^{j-1} + x^{j-2}a + \cdots + xa^{j-2} + a^{j-1}\right| \\
&\leq |a_j| \, |x-a| \left(\left|x^{j-1}\right| + \left|x^{j-2}a\right| + \cdots + \left|xa^{j-2}\right| + \left|a^{j-1}\right|\right) \\
&= |a_j| \, |x-a| \left(|x|^{j-1} + |x|^{j-2}|a| + \cdots + |x| \, |a|^{j-2} + |a|^{j-1}\right) \\
&\leq |a_j| \, |x-a| \left(\underbrace{|x|^{j-1}|a|^{j-1} + \cdots + |x|^{j-1}|a|^{j-1}}_{(j-1) \text{ terms}}\right) \\
&= |a_j| \, |x-a| \, (j-1) \, |x|^{j-1} |a|^{j-1} \\
&\leq |a_j| \, |x-a| \, (j-1)(2|a|)^{j-1} |a|^{j-1} \\
&= |a_j| \, |x-a| \, (j-1)(2a^2)^{j-1}
\end{aligned} \tag{20.B.1}$$

provided that $|x-a| < |a|$. Thus if we defined

$$\delta(\epsilon) = \inf\left\{|a|, \frac{\epsilon}{|a_j| \, (j-1)(2a^2)^{j-1}}\right\},$$

then when $|x-a| < \delta(\epsilon)$, the inequality (20.B.1) holds and we have $|f_j(x) - f_j(a)| < \epsilon$. Hence $f_j(x)$ is continuous on $\mathbf{R}$. Since

$$f(x) = \sum_{j=0}^{n} f_j(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

it follows from Theorem 20.6 that f is continuous on $\mathbf{R}$.

EXERCISE (20.E.). Let f be the function on $\mathbf{R}$ to $\mathbf{R}$ defined by

$$\begin{aligned} f(x) &= x, && x \text{ irrational}, \\ &= 1 - x, && x \text{ rational}. \end{aligned}$$

Show that f is continuous at $x = \frac{1}{2}$ and discontinuous elsewhere.

SOLUTION. Let $D(f) = \mathbf{R}$ and let f be defined by

$$\begin{aligned} f(x) &= x, && x \text{ irrational}, \\ &= 1 - x, && x \text{ rational}. \end{aligned}$$

It may be seen that f is continuous at $a = \frac{1}{2}$. Indeed, let $a = \frac{1}{2}$ and let $\epsilon > 0$; then

$$\begin{aligned} f(x) - f(\frac{1}{2}) = & & x - \frac{1}{2}, && x \text{ irrational}, \\ &= 1 - x - \left(1 - \frac{1}{2}\right) = -(x - \frac{1}{2}), && && x \text{ rational}, \end{aligned}$$

so that $\left|f(x) - f(\frac{1}{2})\right| = \left|x - \frac{1}{2}\right|$. Thus if we defined $\delta(\epsilon) = \epsilon$, then when $\left|x - \frac{1}{2}\right| < \delta(\epsilon)$, we have $\left|f(x) - f(\frac{1}{2})\right| < \epsilon$. Hence f is continuous at $\frac{1}{2}$.

We shall show that f is not continuous at all points $a \neq \frac{1}{2}$ by using the Discontinuity Criterion 20.3. In fact, if $a \neq \frac{1}{2}$ is an rational number, let $X = (x_n)$ be a sequence of irrational numbers converging to a. (Theorem 6.10 assures us of the existence of such a sequence.) Since $f(x_n) = x_n$ for all $n \in \mathbf{N}$, the sequence $(f(x_n))$ does not converge to $f(a) = 1 - x$ and f is not continuous at the rational number a. On the other hand, if b is an irrational number, then there exists a sequence $Y = (y_n)$ of rational numbers converging to b. Since $f(y_n) = 1 - y_n$ for all $n \in \mathbf{N}$, the sequence $(f(y_n))$ does not converge to $f(b) = b$ and f is not continuous at the irrational number b. Therefore, f is continuous at $x = \frac{1}{2}$ and discontinuous elsewhere.

EXERCISE (20.F.). Let f be continuous on $\mathbf{R}$ to $\mathbf{R}$. Show that if $f(x) = 0$ for rational x then $f(x) = 0$ for all x in $\mathbf{R}$.

SOLUTION. Suppose that f is continuous on $\mathbf{R}$ to $\mathbf{R}$ such that $f(x) = 0$ for rational x. Let x belong to $\mathbf{R}$. For each $n \in \mathbf{N}$, choose a rational number x_n in the interval $(x - \frac{1}{n}, x)$ (Theorem 6.10 assures us of the existence of such a rational number.) Then $|x - x_n| < 1/n$, so $\lim(x_n) = x$. Therefore by Theorem 20.2, $f(x) = \lim(f(x_n)) = \lim 0 = 0$.

Exercise (20.G.). Let f and g be continuous on $\mathbf{R}$ to $\mathbf{R}$. Is it true that $f(x) = g(x)$ for $x \in \mathbf{R}$ if and only if $f(y) = g(y)$ for all rational numbers y in $\mathbf{R}$?

Solution. If $f(x) = g(x)$ for $x \in \mathbf{R}$, then $f(y) = g(y)$ for all rational numbers y in $\mathbf{R}$.

Conversely, suppose that f and g are continuous on $\mathbf{R}$ to $\mathbf{R}$ such that $f(y) = g(y)$ for rational y. Let $h(x) = f(x) - g(x)$. It follows from Theorem 20.6 that h is continuous on $\mathbf{R}$ to $\mathbf{R}$. Since $f(y) = g(y)$ for rational y, $h(y) = 0$ for rational y. Therefore by Exercise 20.F, $h(x) = 0$ for $x \in \mathbf{R}$, so that $f(x) = g(x)$ for $x \in \mathbf{R}$.

Exercise (20.H.). Use the inequality $|\sin x| \leq |x|$ for $x \in \mathbf{R}$ to show that the sine function is continuous at $x \neq 0$. Use this fact, together with the identity

$$\sin x - \sin u = 2\sin\left[\frac{1}{2}(x-u)\right]\cos\left[\frac{1}{2}(x+u)\right],$$

to prove that the sine function is continuous ar anypoint of $\mathbf{R}$.

Solution. We shall show that the sine function sin is continuous at 0. To do so we make use of the following inequality of the sine function. For $x \in \mathbf{R}$ we have:

$$|\sin x| \leq |x|\,.$$

Hence let $a = 0$ and let $\epsilon > 0$; then $|\sin x - \sin 0| = |\sin x| \leq |x|$. Thus if we defined $\delta(\epsilon) = \epsilon$, then when $|x - 0| < \delta(\epsilon)$, we have $|\sin x - \sin 0| < \epsilon$. Hence sin is continuous at 0.

We shall show that the sine function sin is continuous at any point on $\mathbf{R}\backslash\{0\}$. To do so we make use of the following inequalities and identity of the sine function. For $x, u \in \mathbf{R}$ we have:

$$|\sin x| \leq |x|\,, \qquad |\cos x| \leq 1,$$

$$\sin x - \sin u = 2\sin\left[\frac{1}{2}(x-u)\right]\cos\left[\frac{1}{2}(x+u)\right].$$

Hence let $a \neq 0$ and let $\epsilon > 0$; then $|\sin x - \sin a| \leq 2 \cdot \frac{1}{2}|x-a| \cdot 1 = |x-a|$. Thus if we defined $\delta(\epsilon) = \epsilon$, then when $|x - 0| < \delta(\epsilon)$, we have $|\sin x - \sin 0| < \epsilon$. Hence sin is continuous at at any point on $\mathbf{R}\backslash\{0\}$. Therefore, the sine function sin is continuous at any point on $\mathbf{R}$.

REMARK. The cosine function is continuous on $\mathbf{R}$. We make use of the following properties of the sine and cosine functions. For all $x, u \in \mathbf{R}$ we have:

$$|\sin x| \leq |x|\,, \quad |\cos x| \leq 1,$$

$$\sin x - \sin u = 2 \sin \left[\frac{1}{2}(x-u)\right] \cos \left[\frac{1}{2}(x+u)\right].$$

Hence if $a \in \mathbf{R}$, then we have

$$|\cos x - \cos a| \leq 2 \cdot 1 \cdot \frac{1}{2}\,|x-a| = |x-a|\,.$$

Therefore cos is continuous at a. Since $a \in \mathbf{R}$ is arbitrary, it follows that cos is continuous on $\mathbf{R}$. (Alternatively, we could use the relation $\cos x = \sin(x + \pi/2)$.)

EXERCISE (20.I.). Using the results of the preceding exercise, show that the function g, defined on $\mathbf{R}$ to $\mathbf{R}$ by

$$\begin{aligned} g(x) &= x \sin(1/x), && x \neq 0, \\ &= 0, && x = 0, \end{aligned}$$

is continuous at every point. Sketch a graph of this function.

SOLUTION. Let $D(g) = \mathbf{R}$ and let g be defined by

$$\begin{aligned} g(x) &= x \sin(1/x), && x \neq 0, \\ &= 0, && x = 0. \end{aligned}$$

Let $a = 0$ and let $\epsilon > 0$; then $|g(x) - g(0)| = |x \sin(1/x) - 0| = |x \sin(1/x)| = |x|\,|\sin(1/x)|$. We want to obtain a bound for $|\sin(1/x)|$ on a neighborhood of 0. In fact, $|\sin y| \leq 1$ for any real number y. Hence,

$$|g(x) - g(0)| \leq |x|\,. \tag{20.I.1}$$

Thus if we define $\delta(\epsilon) = \epsilon$, then when $|x - 0| < \delta(\epsilon)$, the inequality (20.I.1) holds and we have $|g(x) - g(0)| < \epsilon$. Hence g is continuous at 0.

Let $a \neq 0$ belong to $\mathbf{R}$. It was seen in Example 20.5(b) that x is continuous at any point of $\mathbf{R}$, and it was seen in Exercise 20.H that $\sin(1/x)$ is continuous at any point of $\mathbf{R}\backslash\{0\}$, then it follows from Theorem 20.6 that g is continuous at any point of $\mathbf{R}\backslash\{0\}$.

Therefore, g is continuous at every point.

Exercise (20.N.). Let $g\colon \mathbf{R} \to \mathbf{R}$ satisfy the relation

$$g(x+y) = g(x)g(y) \qquad \text{for } x, y \in \mathbf{R}.$$

Show that, if g is continuous at $x = 0$, then g is continuous at every point. Also, if $g(a) = 0$ for some $a \in \mathbf{R}$, then $g(x) = 0$ for all $x \in \mathbf{R}$.

Solution. For the first part, let $g\colon \mathbf{R} \to \mathbf{R}$ satisfy the relation

$$g(x+y) = g(x)g(y) \qquad \text{for } x, y \in \mathbf{R}.$$

Then

$$g(x) = g(x+0) = g(x)g(0) \qquad \text{for } x \in \mathbf{R}.$$

Hence, either $g(0) = 0$, in which case $g(x) = 0$ for all x in $\mathbf{R}$, or $g(0) = 1$, in which case $g(a+h) - g(a) = g(a)\{g(h) - g(0)\}$.

Suppose that g is continuous at $x = 0$ and that $\epsilon > 0$, then there exists a number $\delta(\epsilon) > 0$ such that if $x \in \mathbf{R}$ and $|x - 0| < \delta(\epsilon)$, then $|g(x) - g(0)| < \epsilon$. We will consider two cases, depending on whether $g(0) = 0$, or $g(0) = 1$.

Case 1: Suppose $g(0) = 0$. Then $g(x) = g(x+0) = g(x)g(0) = 0$ for $x \in \mathbf{R}$. Therefore by Example 20.5(a), g is continous at every point of $\mathbf{R}$.

Case 2: Suppose $g(0) = 1$. Let a belong to $\mathbf{R}$ and let $\epsilon > 0$; then

$$\begin{aligned} |g(x) - g(a)| &= |g(a+h) - g(a)| \\ &= |g(a)\{g(h) - g(0)\}| \\ &= |g(a)|\,|g(h) - g(0)| \end{aligned}$$

for some real number h. We wish to make the above expression less than ϵ by making $|g(h) - g(0)|$ sufficiently small. However, the continuity of g at 0 assure us this fact. Therefore, g is continuous at every point of $\mathbf{R}$.

For the second part, suppose that $g(a) = 0$ for some $a \in \mathbf{R}$. Then $g(x) = g(x - a + a) = g(x-a)g(a) = 0$ for $x \in \mathbf{R}$.

CHAPTER 21

Linear Functions

Exercise (21.I.). Let g be a linear function from $\mathbf{R}^p$ to $\mathbf{R}^q$. Show that g is one-one if and only if $g(x) = 0$ implies that $x = 0$.

Solution. It is important to note that if g is a linear function from $\mathbf{R}^p$ to $\mathbf{R}^q$, then $g(0) = 0$; one can see this from the Definition 21.1 because

$$g(0) = g(0 + 0) = g(0) + g(0).$$

If g is one-one, then we shall show show that $g(x) = 0$ implies that $x = 0$. We proceed by contradiction and suppose that there exists $x \neq 0$ such that $g(x) = 0$. Thus $g(x) = 0 = g(0)$. This contradicts the fact that g is one-one. Hence $g(x) = 0$ implies that $x = 0$.

Conversely, suppose that $g(x) = 0$ implies that $x = 0$. Then if $g(x) = g(y)$, by linearity we have $0 = g(x) - g(y) = g(x - y)$, so that $x - y = 0$, and hence $x = y$. Thus f is one-one.

Exercise (21.J.). If h is a one-one linear function from $\mathbf{R}^p$ onto $\mathbf{R}^p$, show that the inverse h^{-1} is a linear function from $\mathbf{R}^p$ onto $\mathbf{R}^p$.

Solution. We repeat ourselves in order to underscore a point. When h is one-one and onto, there is a uniquely determined inverse function h^{-1} which maps $\mathbf{R}^p$ onto $\mathbf{R}^p$ such that $h^{-1}h$ is the identity function on $\mathbf{R}^p$ (domain), and hh^{-1} is the identity function on $\mathbf{R}^p$ (codomain). What we are proving here is that if a linear function h is one-one and onto, then the inverse h^{-1} is also linear.

Let β_1 and β_2 be vectors in $\mathbf{R}^p$ (codomain) and let a, b in $\mathbf{R}$. We wish to show that

$$h^{-1}(a\beta_1 + b\beta_2) = ah^{-1}(\beta_1) + bh^{-1}(\beta_2).$$

Let $\alpha_i = h^{-1}(\beta_i)$, $i = 1, 2$, that is, let a_i be the unique vector in $\mathbf{R}^p$ (domain) such that $h(\alpha_i) = \beta_i$. Since h is linear,

$$\begin{aligned} h(a\alpha_1 + b\alpha_2) &= ah(\alpha_1) + bh(\alpha_2) \\ &= a\beta_1 + b\beta_2. \end{aligned}$$

Thus $a\alpha_1 + b\alpha_2$ is the unique vector in $\mathbf{R}^p$ (codomain) which is sent by h into $a\beta_1 + b\beta_2$, and so

$$\begin{aligned} h^{-1}(a\beta_1 + b\beta_2) &= a\alpha_1 + b\alpha_2 \\ &= a\left[h^{-1}(\beta_1)\right] + b\left[h^{-1}(\beta_2)\right] \end{aligned}$$

and h^{-1} is linear.

Exercise (21.K.). Show that the sum and the composition of two linear functions are linear functions.

Solution. Let f and g be linear functions from $\mathbf{R}^p$ into $\mathbf{R}^q$. We shall show that the function $(f + g)$ defined by

$$(f + g)(x) = f(x) + g(x)$$

is a linear function from $\mathbf{R}^p$ into $\mathbf{R}^q$.

Suppose f and g be linear functions from $\mathbf{R}^p$ into $\mathbf{R}^q$ and that we define $f + g$ as above. Then

$$\begin{aligned} (f + g)(ax + by) &= f(ax + by) + g(ax + by) \\ &= af(x) + bf(y) + ag(x) + bg(y) \\ &= a\left[f(x) + g(x)\right] + b\left[f(y) + g(y)\right] \\ &= a(f + g)(x) + b(f + g)(y) \end{aligned}$$

for all a, b in $\mathbf{R}$ and x, y in $\mathbf{R}^p$, which shows that $(f+g)$ is a linear function.

REMARK. If c is any natrual number, the function (cf) defined by

$$(cf)(x) = c(f(x))$$

is also a linear function from $\mathbf{R}^p$ into $\mathbf{R}^q$. Similarly to the argument above,

$$\begin{aligned}
(cf)(ax+by) &= c\,[f(ax+by)] \\
&= c\,[af(x)+bf(y)] \\
&= caf(x)+cbf(y) \\
&= a\,[cf(x)]+b\,[cf(y)] \\
&= a\,[(cf)(x)]\,b\,[(cf)(y)]
\end{aligned}$$

for all a, b in $\mathbf{R}$ and x, y in $\mathbf{R}^p$, which shows that (cf) is a linear function.

Let f be a linear function from $\mathbf{R}^p$ into $\mathbf{R}^q$ and g a linear function from $\mathbf{R}^q$ into $\mathbf{R}^s$. We shall show that the composed function $g \circ f$ defined by $(g \circ f)(x) = g(f(x))$ is a linear function from $\mathbf{R}^p$ into $\mathbf{R}^s$. In fact,

$$\begin{aligned}
(g \circ f)(ax+by) &= g\,[f(ax+by)] \\
&= g\,[af(x)+bf(y)] \\
&= a\,\{g\,[f(x)]\}+b\,\{g\,[f(y)]\} \\
&= a(g \circ f)(x)+b(g \circ f)(y)
\end{aligned}$$

for all a, b in $\mathbf{R}$ and x, y in $\mathbf{R}^p$, which shows that $(g \circ f)$ is a linear function.

Exercise (21.L.). If f is a linear map on $\mathbf{R}^p$ to $\mathbf{R}^q$, defined

$$\|f\|_{pq} = \sup\{\|f(x)\| : x \in \mathbf{R}^p, \|x\| \leq 1\}.$$

Show that the mapping $f \mapsto \|f\|_{pq}$ defines a norm on the vector space $\mathscr{L}(\mathbf{R}^p, \mathbf{R}^q)$ of all linear functions on $\mathbf{R}^p$ to $\mathbf{R}^q$. Show that $\|f(x)\| \leq \|f\|_{pq}\,\|x\|$ for all $x \in \mathbf{R}^p$.

Solution. (i) Let f be a linear map on $\mathbf{R}^p$ to $\mathbf{R}^q$. Since $\|f(x)\| \geq 0$ for $x \in \mathbf{R}^p$, $\|x\| \leq 1$, it follows that $\sup\{\|f(x)\| : x \in \mathbf{R}^p, \|x\| \leq 1\} \geq 0$. Thus $\|f\|_{pq} \geq 0$ for all $f \in \mathscr{L}(\mathbf{R}^p, \mathbf{R}^q)$, showing that the definition above satisfies property (i) in 8.5.

(ii) If $\|f\|_{pq} = 0$, then$\|f(x)\| = 0$ for $x \in \mathbf{R}^p$, $\|x\| \leq 1$, whence it follows that $\left\|f(x\frac{\|x\|}{\|x\|})\right\| = 0$ for $x \in \mathbf{R}^p$, $\|x\| \leq 1$. Since f is a linear map on $\mathbf{R}^p$ to $\mathbf{R}^q$, it follows that $\|x\|\left\|f\left(\frac{x}{\|x\|}\right)\right\| = 0$ for $x \in \mathbf{R}^p$, $\|x\| \leq 1$. Thus f is the zero function from $\mathbf{R}^p$ to $\mathbf{R}^q$. Conversely, suppose f be the zero function from $\mathbf{R}^p$ to $\mathbf{R}^q$. Then $\|f(x)\| = 0$ for $x \in \mathbf{R}^p$, $\|x\| \leq 1$, whence it follows

that $\sup\{\|f(x)\| : x \in \mathbf{R}^p, \|x\| \le 1\} = 0$. Thus $\|f\|_{pq} = 0$. This shows that the definition above satisfies property (ii) in 8.5.

(iii) For all $a \in \mathbf{R}$ and $f \in (x_1, x_2, \ldots, x_p) \in \mathscr{L}(\mathbf{R}^p, \mathbf{R}^q)$, we have

$$\begin{aligned}
\|af\|_{pq} &= \sup\{\|af(x)\| : x \in \mathbf{R}^p, \|x\| \le 1\} \\
&= \sup\{|a|\,\|f(x)\| : x \in \mathbf{R}^p, \|x\| \le 1\} \\
&= |a| \sup\{\|f(x)\| : x \in \mathbf{R}^p, \|x\| \le 1\} \\
&= |a|\,\|f\|_{pq}\,,
\end{aligned}$$

showing that the definition above satisfies property (iii) in 8.5.

(iv) For all f and g in $\mathscr{L}(\mathbf{R}^p, \mathbf{R}^q)$, we have

$$\begin{aligned}
\|f+g\|_{pq} &= \sup\{\|f(x)+g(x)\| : x \in \mathbf{R}^p, \|x\| \le 1\} \\
&\le \sup\{\|f(x)\| + \|g(x)\| : x \in \mathbf{R}^p, \|x\| \le 1\} \\
&\le \sup\{\|f(x)\| : x \in \mathbf{R}^p, \|x\| \le 1\} + \sup\{\|g(x)\| : x \in \mathbf{R}^p, \|x\| \le 1\} \\
&= \|f\|_{pq} + \|g\|_{pq}\,,
\end{aligned}$$

showing that the definition above satisfies property (iv) in 8.5.

Therefore the definition above satisfies the properties in 8.5.

Next, if $x = 0$, then it is important to note that if f is a linear function from $\mathbf{R}^p$ to $\mathbf{R}^q$, then $f(0) = 0$; one can see this from the Definition 21.1 because

$$f(0) = f(0+0) = f(0) + f(0).$$

Thus $\|f(0)\| = 0 = \|f\|_{pq}\,\|0\|$, and the given inequality holds. If $x \ne 0$, then

$$\begin{aligned}
\|f(x)\| &= \left\| f\left(x \frac{\|x\|}{\|x\|} \right) \right\| \\
&= \left\| f\left(\frac{x}{\|x\|} \right) \right\| \|x\| \\
&\le \sup\{f(y) : y \in \mathbf{R}^p, \|y\| \le 1\}\,\|x\| \\
&= \|f\|_{pq}\,\|x\|\,,
\end{aligned}$$

and the given inequality holds.

CHAPTER 22

Global Properties of Continuous Functions

Exercise (22.B.). Let $h\colon \mathbf{R} \to \mathbf{R}$ be defined by

$$\begin{aligned} h(x) &= 1, \qquad && 0 \le x \le 1 \\ &= 0, && \text{otherwise.} \end{aligned}$$

Exhibit an open set G such that $h^{-1}(G)$ is not open, and a closed set F such that $h^{-1}(F)$ is not closed.

Solution. If G is the open set $G = (0, 2)$, then $h^{-1}(G) = h^{-1}(1) = [0, 1]$, which is not open in $\mathbf{R}$. Similarly, if F is the closed set $F = \{0\}$, then $h^{-1}(F) = h^{-1}(0) = (-\infty, 0) \cup (1, +\infty)$, which is not closed in $\mathbf{R}$.

Exercise (22.C.). If f is bounded and continuous on $\mathbf{R}^p$ to $\mathbf{R}$ and if $f(x_0) > 0$, show that f is strictly positive on some neighborhood of x_0. Does the same conclusion hold if f is merely continuous at x_0.

Solution. Let $V = \{y \in \mathbf{R} : y > 0\}$. Then $f(x_0) \in V$. Since $f(x_0) > 0$ and V is an open set in $\mathbf{R}$, it follows that $V = \{y \in \mathbf{R} : y > 0\}$ is a neigborhood of $f(x_0)$. Since the function f is continuous at x_0, it follows from Corollary 22.2(b) that there is an open set U containing x_0 in $\mathbf{R}^q$ such that $f(U) \subset V$. Thus f is strictly positive on the neighborhood U of x_0.

Let $f\colon \mathbf{R} \to \mathbf{R}$ be defined by

$$\begin{aligned} f(x) &= 0, \qquad && x < 0 \\ &= 1, && x \ge 0. \end{aligned}$$

Then $f(0) = 1 > 0$ but there does not exist any neighborhood of x_0 such that $f(x)$ is positive on that neighborhood.

Exercise (22.F.). A subset $D \subseteq \mathbf{R}^p$ is disconnected if and only if there exists a continuous function $f\colon D \to \mathbf{R}$ such that $f(D) = \{0, 1\}$.

Solution. If $D \subseteq \mathbf{R}^p$ is disconnected, then there exist two open sets A, B such that $A \cap D$ and $B \cap D$ are disjoint, non-empty, and have union D. Let $f\colon D \to \mathbf{R}$ be defined by

$$\begin{aligned} f(x) &= 0, && \text{if } x \in A \cap D, \\ &= 1, && \text{if } x \in B \cap D. \end{aligned}$$

We shall show that f is continuous. Indeed, let G is open in $\mathbf{R}$. If G is the open set that does not contain both points 0 and 1, then $f^{-1}(G) = \emptyset$, which is open in D. If G is the open set that contains both points 0 and 1, then $f^{-1}(G) = D$, which is open in D. If G is the open set that contains point 0 but does not contain point 1, then $f^{-1}(G) = A \cap D$, which is open in D. Similarly, if G is the open set that contains point 1 but does not contain point 0, then $f^{-1}(G) = B \cap D$, which is open in D. Therefore by Theorem 22.1, there exists a continuous function $f\colon D \to \mathbf{R}$ such that $f(D) = \{0, 1\}$.

Conversely, suppose there exists a continuous function $f\colon D \to \mathbf{R}$ such that $f(D) = \{0, 1\}$. We shall prove that the subset $D \subseteq \mathbf{R}^p$ is disconnected by contradiction and suppose that D is connected. Thus by Theorem 22.3, $f(D)$ is connected. This contradicts the fact that $f(D) = \{0, 1\}$ is disconnected in $\mathbf{R}$. Therefore, D is disconneted.

Exercise (22.H.). Let f, g_1, g_2 be related by the formulas in the preceeding exercise. Show that from the continuity of g_1 and g_2 at $t = 0$ one cannot prove the continuity of f at $(0, 0)$.

Solution. Let $f(s, t) = 0$ if $st = 0$ and $f(s, t) = 1$ if $st \neq 0$. Then $g_1(t) = f(t, 0) = 0$ and $g_2(t) = f(0, t) = 0$ are continuous at $t = 0$ but f is not continuous at $(0, 0)$.

Exercise (22.K.). Give an example of a bounded and continuous function g on $\mathbf{R}$ to $\mathbf{R}$ which does not take on either of the numbers $\sup\{f(x) : x \in \mathbf{R}\}$ or $\inf\{f(x) : x \in \mathbf{R}\}$

Solution. For an example, let $f\colon \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \arctan(x)$. Then $\sup\{f(x) : x \in \mathbf{R}\} = \pi/2$ but the supremum is not attained. Similarly for the infimum.

Exercise (22.O.). Let f be a continuous function on $\mathbf{R}$ to $\mathbf{R}$ which is strictly increasing (in the sense that if $x' < x''$ then $f(x') < f(x'')$). Prove that f is injective and that its inverse function f^{-1} is continuous and strictly increasing.

SOLUTION. Since f is strictly increasing on $\mathbf{R}$, it follows from Thereom 5.4(b) that for all $x \neq y$ in $\mathbf{R}$, then $x < y$ or $x > y$, so that $f(x) < f(y)$ or $f(x) > f(y)$, and hence $f(x) \neq f(y)$. Thus f is injective.

Since f is injective by the above argument, the inverse function $g = f^{-1}$ is defined. We claim that f^{-1} is strictly increasing. Indeed, if $y_1, y_2 \in \mathbf{R}$ with $y_1 < y_2$, then $y_1 = f(x_1)$ and $y_2 = f(x_2)$ for some $x_1, x_2 \in \mathbf{R}$. We must have $x_1 < x_2$; otherwise $x_1 \geq x_2$, which implies that $y_1 = f(x_1) \geq f(x_2) = y_2$, contrary to the hypothesis that $y_1 < y_2$. Therefore we have $f^{-1}(y_1) = x_1 < x_2 = f^{-1}(y_2)$. Since y_1 and y_2 are arbitrary elements of $\mathbf{R}$ with $y_1 < y_2$, we conclude that f^{-1} is strictly increasing on $\mathbf{R}$.

It remains to show that f^{-1} is continuous on $\mathbf{R}$. Let $D(f^{-1}) = \mathbf{R}$. Let y_0 belong to $\mathbf{R}$ and let $\epsilon > 0$; then $\left|f^{-1}(y) - f^{-1}(y_0)\right| = \left|f^{-1}(y) - x_0\right|$ where $f(x_0) = y_0$ for some $x_0 \in \mathbf{R}$. We wish to make the above expression less than ϵ by making $|y - y_0|$ sufficiently small. We note that $x_0 - \epsilon < x_0 < x_0 + \epsilon$, then $f(x_0 - \epsilon) < f(x_0) < f(x_0 + \epsilon)$. Thus if we define

$$\delta(\epsilon) = \inf\{f(x_0) - f(x_0 - \epsilon), f(x_0 + \epsilon) - f(x_0)\},$$

then $f(x_0 - \epsilon) < f(x_0) - \delta(\epsilon)$ and $f(x_0) + \delta(\epsilon) < f(x_0 + \epsilon)$. Hence if $f(x_0) - \delta(\epsilon) < y < f(x_0) + \delta(\epsilon)$, then $f(x_0 - \epsilon) < y < f(x_0 + \epsilon)$. Since f is strictly increasing, so is f^{-1} and therefore $x_0 - \epsilon < f^{-1}(y) < a + \epsilon$. We have shown $\left|f^{-1}(y) - f^{-1}(y_0)\right| < \epsilon$ if $|y - y_0| < \delta(\epsilon)$ which is what we needed to show.

CHAPTER 23

Uniform Continuity and Fixed Points

Exercise (23.A.). Examine each of the functions in Example 20.5 and either show that the function is uniformly continuous on its domain or that it is not.

Solution. (a) Let $D(f) = \mathbf{R}$ and let f be the "constant" function defined to be equal to the real number c for all real numbers x. Then f is uniformly continuous on $\mathbf{R}$; in fact, given $\epsilon > 0$ there is a $\delta(\epsilon) = \epsilon$ such that if x and u belong to $\mathbf{R}$ and $|x-u| < \delta(\epsilon)$, then $|f(x) - f(u)| = 0 < \epsilon$.

(b) Let $D(f) = \mathbf{R}$ and let f be the "identity" function defined by $f(x) = x$. Given $\epsilon > 0$ there is a $\delta(\epsilon) = \epsilon$ such that if x and u belong to $\mathbf{R}$ and $|x-u| < \delta(\epsilon)$, then $|f(x) - f(u)| = |x-u| < \epsilon$. Thus f is uniformly continuous on $\mathbf{R}$.

(c) Let $D(f) = \mathbf{R}$ and let f be the "squaring" function defined by $f(x) = x^2$, $x \in \mathbf{R}$. It was seen in Example 20.5(b) that f is continuous at every point of $\mathbf{R}$. However, f is not uniformly continuous on $\mathbf{R}$. Let $\delta > 0$; according to Corollary 6.7(b) (of the Archimedean Property) there exists a natural number $K(\delta)$ such that $1/K(\delta) < \delta$. Then, if $x = K(\delta) + 1/K(\delta)$ and $u = K(\delta)$ we have

$$|f(x) - f(u)| = \left|f\left(K(\delta) + \frac{1}{K(\delta)}\right) - f\left(K(\delta)\right)\right| = \left|\left[K(\delta) + \frac{1}{K(\delta)}\right]^2 - [K(\delta)]^2\right| = \left|2 + \frac{1}{[K(\delta)]^2}\right| > 2,$$

provided that $|x-u| = |[K(\delta) + 1/K(\delta)] - K(\delta)| = 1/K(\delta) < \delta$. Therefore we infer that the function f does not converges uniformly on $\mathbf{R}$.

(d) Similar argument to (c)

(e) Consider $D(f) = \{x \in \mathbf{R} : x \neq 0\} = (-\infty, 0) \cup (0, +\infty)$ and let f be defined by $f(x) = 1/x$, $x \in D(f)$. It was seen in Example 20.5(e) that f is continuous at every point of $D(f)$. However, f is not uniformly continuous on $D(f)$. We first wish to show that f is not uniformly continuous on $(0, +\infty)$. Let $\delta > 0$; according to Corollary 6.7(b) (of the Archimedean

Property) there exists a natural number $K(\delta)$ such that $1/K(\delta) < \delta$. Then, if $x = 1/K(\delta)$ and $u = 1/2K(\delta)$ we have

$$|f(x) - f(u)| = \left| f\left(\frac{1}{K(\delta)}\right) - f\left(\frac{1}{2K(\delta)}\right) \right| = |K(\delta) - 2K(\delta)| = |-K(\delta)| = K(\delta) \geq 1,$$

provided that $|x - u| = |[1/K(\delta)] - [1/2K(\delta)]| = 1/2K(\delta) < 1/K(\delta) < \delta$. Thus we infer that the function f does not converges uniformly on $(0, +\infty)$. A similar argument show that the function f does not converges uniformly on $(-\infty, 0)$. Therefore the function f does not converges uniformly on $D(f)$.

(f) Let f be defined for $D(f) = \mathbf{R}$ by

$$\begin{aligned} f(x) &= 0, \qquad && x \leq 0, \\ &= 1, && x > 0. \end{aligned}$$

It was seen in Example 20.5(f) that f is continuous at all points $u \neq 0$ and not continuous at 0, so that f is not uniformly continuous on $\mathbf{R}$.

(g) Let $D(f) = \mathbf{R}$ and let f be Dirichlet's discontinuous function defined by

$$\begin{aligned} f(x) &= 1, \qquad && \text{if } x \text{ is rational}, \\ &= 1, && \text{if } x \text{ is irrational}. \end{aligned}$$

It was seen in Example 20.5(g) that f is not continuous at any point, so that f is not uniformly continuous on $\mathbf{R}$.

(h) Let $D(f) = \{x \in \mathbf{R} : x > 0\}$ For any irrational number $x > 0$, we define $f(x) = 0$; for a rational number of the form m/n, with the natural numbers m, n having no common factor except 1, we define $f(m/n) = 1/n$. It was seen in Example 20.5(h) that f is continuous at every irrational number in $D(f)$ and discontinuous at every rational number in $D(f)$, so that f is not uniformly continuous on $D(f)$.

(i) This times, let $D(f) = \mathbf{R}^2$ and let f be the function on $\mathbf{R}^2$ with values in $\mathbf{R}^2$ defined by

$$f(x, y) = (2x + y, x - 3y).$$

Let (a, b) be a fixed point in $\mathbf{R}^2$; we shall show that f is uniformly continuous on $\mathbf{R}^2$. To do this, we need to show that we can make the expression

$$\|f(x, y) - f(a, b)\| = \left\{(sx + y - 2a - b)^2 + (x - 3y - a + 3b)^2\right\}^{1/2}$$

arbitrarily small by choosing (x, y) sufficiently close to (a, b). Since $\left\{p^2 + q^2\right\}^{1/2} \leq \sqrt{2} \sup \{|p|, |q|\}$, it is evidently enough to show that we can make the terms

$$|2x + y - 2a - b|, \qquad\qquad |x - 3y - a + 3b|,$$

arbitrarily small by choosing (x, y) sufficiently close to (a, b). In fact, by the Triangle Inequality,

$$|2x + y - 2a - b| = |2(x - a) + (y - b)| \leq 2|x - a| + |y - b|.$$

Now $|x - a| \leq \left\{(x - a)^2 + (y - b)^2\right\}^{1/2} = \|(x, y) - (a, b)\|$, and similarly for $|y - b|$; hence we have

$$|2x + y - 2a - b| \leq 2\|(x, y) - (a, b)\|.$$

Similarly,

$$|x - 3y - a + 3b| \leq |x - a| + 3|y - b| \leq 4\|(x, y) - (a, b)\|.$$

Therefore, if $\epsilon > 0$, we can take $\delta(\epsilon) = \epsilon/(4\sqrt{2})$ which is independent of the choice of (a, b) for all points $(a, b) \in \mathbf{R}^2$ and be certain that if $\|(x, y) - (a, b)\| < \delta(\epsilon)$, then $\|f(x, y) - f(a, b)\| < \epsilon$, although a larger value of δ can be attained by a more refined analysis (for example, by using the Schwarz Inequality 8.7).

(j) Again let $D(f) = \mathbf{R}^2$ and let f be defined by

$$f(x, y) = (x^2 + y^2, 2xy)$$

It was seen in Example 20.5(j) that f is continuous on $\mathbf{R}^2$. However, f is not uniformly continuous on $\mathbf{R}^2$. Let $\delta > 0$; according to Corollary 6.7(b) (of the Archimedean Property) there exists a natural number $K(\delta)$ such that $1/K(\delta) < \delta$. Then, if $(x, y) = (K(\delta) + 1/K(\delta), 0)$ and $(a, b) = (K(\delta), 0)$ we have

$$\begin{aligned}
\|f(x,y)-f(a,b)\| &= \left\|f\left(K(\delta)+\frac{1}{K(\delta)},0\right)-f\left(K(\delta),0\right)\right\| \\
&= \left\{\left[\left(K(\delta)+\frac{1}{K(\delta)}\right)^2+0^2-(K(\delta))^2-0^2\right]^2+\left[2\left(K(\delta)+\frac{1}{K(\delta)}\right)0-2\left(K(\epsilon)\right)0\right]^2\right\}^{1/2} \\
&= \left\{\left[\left(K(\delta)+\frac{1}{K(\delta)}\right)^2-(K(\delta))^2\right]^2\right\}^{1/2} \\
&= \left|\left(K(\delta)+\frac{1}{K(\delta)}\right)^2-(K(\delta))^2\right| \\
&= \left|2+\frac{1}{(K(\delta))^2}\right| \\
&> 2,
\end{aligned}$$

provided that

$$\begin{aligned}
\|(x,y)-(a,b)\| &= \left\|\left(K(\delta)+\frac{1}{K(\delta)},0\right)-(K(\delta),0)\right\| \\
&= \left\{\left(K(\delta)+\frac{1}{K(\delta)}-K(\delta)\right)^2+(0-0)^2\right\}^{1/2} \\
&= \left\{\left(K(\delta)+\frac{1}{K(\delta)}-K(\delta)\right)^2\right\}^{1/2} \\
&= \left|K(\delta)+\frac{1}{K(\delta)}-K(\delta)\right| \\
&= \frac{1}{K(\delta)} \\
&< \epsilon
\end{aligned}$$

Therefore we infer that the function f does not converges uniformly on $\mathbf{R}^2$.

Exercise (23.C.). If B is bounded in $\mathbf{R}^p$ and $f\colon B \to \mathbf{R}^q$ is uniformly continuous, show that f is bounded on B. Show that this conclusion fails if B is not bounded in $\mathbf{R}^p$.

Solution (1). Since the set B is bounded, we may enclosed it in a closed cell I_1 in $\mathbf{R}^p$. For examle, we may take $I_1 = \{(x_1, \ldots, x_p) : |x_k| \leq r, k = 1, \ldots, p\}$ for suitably large $r > 0$. Suppose that f is uniformly continuous on the bounded set B. According to Defintition 23.1, given

$\epsilon = 1$ and u in B there is a number $\delta(1) > 0$ such that if $x \in K$ and $\|x - u\| < \delta(1)$, then $\|f(x) - f(u)\| < \delta(1)$. We divide I_1 into a finite number of closed cells whose length of the largest side does not Since f is uniformly continuous, take $\epsilon = 1$ there exists $\delta > 0$ such that I_1 be a closed cell containing B. We divide I_1 into 2^p closed cells by bisecting each of its sides. We note that if $I_1 = [a_1, b_1] \times \cdots \times [a_p, b_p]$ with $a_k < b_k$, and if $l(I_1) = \sup\{b_1 - a_1, \ldots, b_p - a_p\}$, then $l(I_1) > 0$ is the length of the largest side of I_1.

Solution (2). We assume that B is bounded and f is uniformly continuous on B and shall show that $f(B)$ is bounded in $\mathbf{R}^q$. If $f(B)$ is not bounded, for each $n \in \mathbf{N}$ there exists a point x_n in B with $\|f(x_n)\| \geq n$. Since B is bounded, the sequence $X = (x_n)$ is bounded; hence it follows from the Bolzano-Weierstrass Theorem 16.4 that there is a subsequence $X' = (x_{n_j})$ of X which is a convergent sequence in $\mathbf{R}^p$. It was seen in Lemma 16.7 that the sequence X' is a Cauchy sequence. According to Exercise 23.H, the sequence $(f(x_{n_j}))$ is a Cauchy sequence in $\mathbf{R}^q$. Hence it follows from the Cauchy Convergence Criterion 16.10 that $(f(x_{n_j}))$ is a convergent sequence in $\mathbf{R}^q$, and hence is bounded. This contradicts the fact that $(f(x_{n_j}))$ is not bounded.

Solution (3). If B is bounded, then the closure $B^-$is also bounded. (Why?) Since f is uniformly continuous on B, its extension $\tilde{f}$ must be continuous on B^-. But B^- is a compact set, so $\tilde{f}$ is continuous on a compact set. By Preservation of Compactness 22.5, the image of $\tilde{f}$ is a compact set, and hence is bounded. Therefore f is bounded on B.

Exercise (23.D.). Show that the functions, defined for $x \in \mathbf{R}$ by

$$f(x) = \frac{1}{1 + x^2}, \qquad\qquad g(x) = \sin x,$$

are uniformly continuous on $\mathbf{R}$.

Solution. (a) Let $D(f) = \mathbf{R}$ and let f be defined by $f(x) = 1/(1+x^2)$, $x \in \mathbf{R}$. If u belong to $\mathbf{R}$ and let $\epsilon > 0$; then

$$
\begin{aligned}
|f(x)-f(u)| &= \left|\frac{1}{1+x^2}-\frac{1}{1+u^2}\right| \\
&= \left|\frac{1+u^2-1-x^2}{(1+x^2)(1+u^2)}\right| \\
&= \frac{\left|u^2-x^2\right|}{(1+x^2)(1+u^2)} \\
&= \frac{|x-u|\,|x+u|}{(1+x^2)(1+u^2)}.
\end{aligned}
$$

Using the Triangle Inequality, we obtain

$$|f(x)-f(u)| \le |x-u|\,\frac{|x|+|u|}{(1+x^2)(1+u^2)}.$$

If $x \ge 1$, we have $|x| \le |x|^2$ and $0 < |x| / \left[(1+x^2)(1+u^2)\right] \le 1$. If $x < 1$, the same inequality holds. We also get the same inequality if we exchange x and u. Hence

$$|f(x)-f(u)| \le 2\,|x-u|\,. \tag{23.D.1}$$

Thus if we define $\delta(\epsilon) = \epsilon/2$ which is independent of the choice of u for all values of u, then when $|x-u| < \delta(\epsilon)$, the inequality (23.D.1) holds and we have $|f(x)-f(a)| < \epsilon$.

Exercise (23.F.). Show that the following functions are not uniformly continuous on their domains.

(a) $f(x) = 1/x^2$, $D(f) = \{x \in \mathbf{R} : x > 0\}$,
(b) $g(x) = \tan x$, $D(g) = \{x \in \mathbf{R} : 0 \le x < \pi/2\}$,
(c) $h(x) = e^x$, $D(h) = \mathbf{R}$,
(d) $k(x) = \sin(1/x)$, $D(k) = \{x \in \mathbf{R} : x > 0\}$.

Solution. (a) Let $D(f) = \{x \in \mathbf{R} : x > 0\}$ and let f be defined by $f(x) = 1/x^2$. If $\epsilon_0 = 2$, choose two sequences $X = (x_n)$ and $Y = (y_n)$ in $D(f)$ are such that $x_n = 1/n$ and $y_n = 1/2n$ for $n \in \mathbf{N}$, then

$$|x_n - y_n| = \left|\frac{1}{n} - \frac{1}{2n}\right| = \frac{1}{2n} < \frac{1}{n}$$

and

$$|f(x_n) - f(y_n)| = \left| f\left(\frac{1}{n}\right) - f\left(\frac{1}{2n}\right) \right| = \left|n^2 - (2n)^2\right| = \left|-3n^2\right| = 3n^2 > 2 = \epsilon_0$$

showing that the function f does not converges uniformly on $D(f)$.

EXERCISE (23.G.). A function $g\colon \mathbf{R} \to \mathbf{R}^q$ is **periodic** if there exists a number $p > 0$ such that $g(x + p) = g(x)$ for all $x \in \mathbf{R}$. Show that a continuous periodic function is bounded and uniformly continuous on $\mathbf{R}$.

SOLUTION. Suppose that $g\colon \mathbf{R} \to \mathbf{R}^q$ is periodic. By the definition of a periodic function there exists a number $p > 0$ such that $g(x + p) = g(x)$ for all $x \in \mathbf{R}$. Let $M = \sup\{\|g(x)\| : 0 \le x \le p\}$. If $x \in \mathbf{R}$, then there exists $k \in \mathbf{Z}$ such that $0 \le x + kp \le p$. Hence $\|f(x)\| = \|f(x + kp)\| \le M$, so that f is bounded.

Let $D(g') = [0, p]$ and let $g'\colon [0, p] \to \mathbf{R}^q$ be defined by $g'(x) = g(x)$, $x \in [0, p]$. Since g is continuous on $\mathbf{R}$, so g' is continuous on $[0, p]$ by the defintition of g'. Moreover, since $[0, p]$ is compact in $\mathbf{R}$, it follows from Uniform Continuity Theorem 23.3 that g' is uniformly continous on $[0, p]$. Thus for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that if y and u belong to $[0, p]$ and $|y - u| < \delta(\epsilon)$, then $\|g'(y) - g'(u)\| < \epsilon$. If x_1 and x_2 are real numbers with $|x_1 - x_2| < \delta(\epsilon)$, then there exists $k_1, k_2 \in \mathbf{Z}$ and $y_1, y_2 \in [0, p]$ such that $x_1 = y_1 + k_1 p$ and $x_2 = y_2 + k_2 p$. Hence if $|x_1 - x_2| < \delta(\epsilon)$, then $|y_1 - y_2| < \delta(\epsilon)$. Thus

$$\begin{aligned} \|g(x_1) - g(x_2)\| &= \|g(y_1 + k_1 p) - g(y_2 + k_2 p)\| \\ &< \|g(y_1) - g(y_2)\| \\ &= \left\|g'(y_1) - g'(y_2)\right\| \\ &< \epsilon. \end{aligned}$$

This proves the uniform continuity of f.

EXERCISE (23.H.). Let f be defined on $D \subseteq \mathbf{R}^p$ to $\mathbf{R}^q$, and suppose that f is uniformly continuous on D. If (x_n) is a Cauchy sequence in D, show that $(f(x_n))$ is a Cauchy sequence in $\mathbf{R}^q$.

SOLUTION. Let (x_n) be a Cauchy sequence in D, and let $\epsilon > 0$ be given. First choose $\delta(\epsilon) > 0$ such that if x, u in D satisfy $\|x - u\| \le \delta(\epsilon)$, then $\|f(x) - f(u)\| \le \epsilon$. Since (x_n) is a Cauchy sequence, there exists $M(\delta(\epsilon))$ such that $\|x_m - x_n\| < \delta(\epsilon)$ for all $m, n \ge M(\delta(\epsilon))$. By the choice

of $\delta(\epsilon)$, this implies that for$m, n \geq M(\delta(\epsilon))$, we have $\|f(x_m) - f(x_n)\| < \epsilon$. Therefore the sequence $(f(x_n))$ is a Cauchy sequence.

Suppose that f is uniformly continuous on D. Given $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that if x and u belong to A and $\|x - u\| \leq \delta(\epsilon)$, then $\|f(x) - f(u)\| \leq \epsilon$. Let (x_n) be a Cauchy sequence in D. Given this $\delta(\epsilon)$ there is a natural number $M(\epsilon)$ such that for all $m, n \geq M(\epsilon)$, then $\|x_m - x_n\| < \delta(\epsilon)$, and thus $\|f(x_m) - f(x_n)\| < \epsilon$. Therefore, $(f(x_n))$ is a Cauchy sequence in $\mathbf{R}^q$.

REMARK. The preceding result gives us an alternative way of seeing that $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$.We note that the sequence given by $x_n = 1/n$ in $(0, 1)$ is a Cauchy sequence, but the image sequence, where $f(x_n) = n$, is not a Cauchy sequence.

CHAPTER 24

Sequences of Contuous Functions

Exercise (24.B.). Give an example of a sequence of everywhere discontinuous functions which converges uniformly to a continuous function.

Solution. Let $D(f) = I = [0,1]$ and let f be Dirichlet's discontinuous function defined by

$$\begin{aligned} f(x) &= 1, && \text{if } x \text{ is rational on } [0,1], \\ &= 0, && \text{if } x \text{ is irrational on } [0,1]. \end{aligned}$$

For $n \in \mathbf{N}$, and $x \in \mathbf{I}$, let

$$\begin{aligned} f_n(x) &= \frac{1}{n}, && \text{if } x \text{ is rational on } \mathbf{I}, \\ &= 0, && \text{if } x \text{ is irrational on } \mathbf{I}. \end{aligned}$$

Then f_n is discontinuous everywhere since f is and $f_n \to 0$, as $n \to \infty$, so that

$$\|f_n\|_{[0,1]} = \frac{1}{n} \to 0.$$

Each function f_n is discontinuous at every point of $\mathbf{I}$ and the limit function, $g(x) = 0$, is continuous at every point of $\mathbf{I}$. Therefore, (f_n) is a sequence of everywhere discontinuous functions that uniformly converges to a continuous function.

Exercise (24.D.). Let (f_n) be a sequence of continuous functions on $D \subseteq \mathbf{R}^p$ to $\mathbf{R}^q$ such that (f_n) converges uniformly to f on D, and let (x_n) be a sequence of elements in D which converges to $x \in D$. Does it follow that $(f_n(x_n))$ converges to $f(x)$?

Solution. Yes. We note that

$$\|f_n(x_k) - f(x_k)\| \leq \|f_n - f\| < \epsilon.$$

Hence

$$\lim_{k\to\infty} \|f_n(x_k) - f(x_k)\| < \epsilon,$$

so that

$$\left\| f_n(\lim_{k\to\infty} x_k) - f(\lim_{k\to\infty} x_k) \right\| < \epsilon$$

showing that

$$\|f_n(x) - f(x)\| < \epsilon.$$

EXERCISE (24.E.). Consider the sequences (f_n) defined on $D = \{x \in \mathbf{R} : x \geq 0\}$ by the following formulas:
(a) $\frac{x^n}{n}$,
(b) $\frac{x^n}{1+x^n}$,
(c) $\frac{x^n}{n+x^n}$,
(d) $\frac{x^{2n}}{1+x^n}$,
(e) $\frac{x^{2n}}{1+x^{2n}}$,
(f) $\frac{x}{n}e^{-x/n}$.
Discuss the convergence and the uniform convergence of these sequences and the continuity of the limit functions. In case of non-uniform convergence on D, consider appropriate intervals in D.

SOLUTION. (b) Let $D = \{x \in \mathbf{R} : x \geq 0\}$ and for each natural number n let f_n be defined by $f_n(x) = x^n/(1 + x^n)$ for $x \in D$ and let f be defined by

$$\begin{aligned} f(x) &= 0, && 0 \leq x < 1, \\ &= \frac{1}{2}, && x = 1, \\ &= 1, && x > 1. \end{aligned}$$

If $0 \leq x < 1$, then

$$0 \leq \frac{x^n}{1+x^n} \leq x^n.$$

By Example 14.8(c), we have $\lim(x^n) = 0$ for $0 \leq x < 1$. Hence it follows from Exercise 15.A that

$$0 \leq \lim(f_n(x)) = \lim(x^n/(1+x^n)) \leq \lim(x^n) = 0$$

for $0 \leq x < 1$ and it is readily seen from Exercise 15.B that $\lim(f_n(x)) = 0$ for $0 \leq x < 1$. If $x = 1$, then the sequence $(f_n(1)) = (1/2)$ converges to $1/2$. If $x > 1$, then

$$\frac{x^n}{1+x^n} = \frac{x^n \frac{1}{x^n}}{(1+x^n)\frac{1}{x^n}} = \frac{1}{\left(\frac{1}{x}\right)^n + 1}.$$

The convergence of $x^n/(1+x^n) \to 1$ follows from the fact that $(1/x)^n \to 0$ for $x > 1$. Therefore, we conclude that (f_n) converges on D to f. However, the sequence (f_n) does not converges uniformly on D to f. Indeed, if $n_k = k$ and $x_k = \left(\frac{1}{2}\right)^{1/k}$, then

$$|f_k(x_k) - f(x_k)| = \left| \frac{\left[\left(\frac{1}{2}\right)^{1/k}\right]^k}{1 + \left[\left(\frac{1}{2}\right)^{1/k}\right]^k} - 0 \right| = \left| \frac{\frac{1}{2}}{1+\frac{1}{2}} \right| = \frac{1}{3},$$

showing that (f_n) does not converge uniformly on D to f. Alternatively, since f is not continuous at the point $x = 1$, it follows that the sequence (f_n) does not converges uniformly on D to f. But the sequence (f_n) converges uniformly on any closed set not containing 1. First, we take $E = [0, a]$ with $a < 1$. Note that

$$\begin{aligned} \|f_n - f\|_E &= \sup\{f_n(x) - f(x) : 0 \leq x \leq a\} \\ &= \sup\{\frac{x^n}{1+x^n} : 0 \leq x \leq a\}. \end{aligned}$$

But since $x^n/(1+x^n) \leq x^n \leq a^n$, we conclude that $\|f_n - f\|_E = a^n$. Hence (f_n) converges uniformly on the interval $[0, a]$ with $a < 1$. Next, we take $F = [b, c]$ with $b > 1$. Note that

$$\begin{aligned} \|f_n - f\|_F &= \sup\{f_n(x) - f(x) : b \leq x < +\infty\} \\ &= \sup\{\frac{x^n}{1+x^n} - 1 : b \leq x < +\infty\} \\ &= \sup\{\frac{1}{1+x^n} : b \leq x < +\infty. \end{aligned}$$

But since $x^n \geq b^n$, we conclude that $\|f_n - f\|_F = 1/(1+b^n)$. Hence (f_n) converges uniformly on the interval $[b, +\infty)$ with $b > 1$.

(c) Let $D = \{x \in \mathbf{R} : x \geq 0\}$ and for each natural number n let f_n be defined by $f_n(x) = x^n/(n + x^n)$ for $x \in D$ and let f be defined by

$$\begin{aligned} f(x) &= 0, && 0 \leq x \leq 1, \\ &= 1, && x > 1. \end{aligned}$$

If $0 \leq x < 1$, then

$$0 \leq \frac{x^n}{n + x^n} \leq x^n.$$

By Example 14.8(c), we have $\lim(x^n) = 0$ for $0 \leq x < 1$. Hence it follows from Exercise 15.A that

$$0 \leq \lim(f_n(x)) = \lim(x^n/(1 + x^n)) \leq \lim(x^n) = 0$$

for $0 \leq x < 1$ and it is readily seen from Exercise 15.B that $\lim(f_n(x)) = 0$ for $0 \leq x < 1$. If $x = 1$, then the sequence $(f_n(1)) = (1/(n + 1))$ converges to 0. If $x > 1$, then

$$\frac{x^n}{n + x^n} = \frac{x^n \frac{1}{x^n}}{(n + x^n)\frac{1}{x^n}} = \frac{1}{\frac{n}{x^n} + 1}.$$

The convergence of $x^n/(n+x^n) \to 1$ follows from the fact that $n/x^n \to 0$ for $x > 1$. Therefore, we conclude that (f_n) converges on D to f. However, the sequence (f_n) does not converges uniformly on D to f. Indeed, if $n_k = k$ and $x_k = (2)^{1/k}$, then

$$|f_k(x_k) - f(x_k)| = \left| \frac{\left[(2)^{1/k}\right]^k}{k + \left[(2)^{1/k}\right]^k} - 1 \right| = \left| \frac{2}{k+2} - 1 \right| = 1 - \frac{2}{k+2} \geq 1 - \frac{2}{1+2} = \frac{1}{3},$$

showing that (f_n) does not converge uniformly on D to f. Alternatively, since f is not continuous at the point $x = 1$, it follows that the sequence (f_n) does not converges uniformly on D to f. But the sequence (f_n) converges uniformly on $[0, 1]$ or on $[c, +\infty)$ with $c > 1$. First, we take $E = [0, 1]$. Let $\epsilon > 0$; according to Corollary 6.7(b) (of the Archimedean Property) there exists a natural number $K(\epsilon)$ such that $1/K(\epsilon) < \epsilon$. Then, if $n \geq K(\epsilon)$ we have

$$|f_n(x) - f(x)| = \left| \frac{x^n}{n + x^n} - 0 \right| \leq \frac{x^n}{n} \leq \frac{1}{n} \leq \frac{1}{K(\epsilon)} < \epsilon.$$

Since $K(\epsilon) > 0$ depending on ϵ but not on $x \in E$ this proves that (f_n) conveges unifomly on E to f. Next, taking $F = [c, +\infty)$ with $c > 1$. Note that

$$\begin{aligned} \|f_n - f\|_F &= \sup\{f_n(x) - f(x) : c \leq x < +\infty\} \\ &= \sup\{\frac{x^n}{n + x^n} - 1 : c \leq x < +\infty\} \\ &= \sup\{-\frac{n}{n + x^n} : c \leq x < +\infty\}. \end{aligned}$$

But since $x^n \geq c^n$, we conclude that $\|f_n - f\|_F = -n/(n + c^n)$. Hence (f_n) converges uniformly on the interval $[b, +\infty)$ with $b > 1$.

(d) Let $D = \{x \in \mathbf{R} : x \geq 0\}$ and for each natural number n let f_n be defined by $f_n(x) = x^{2n}/(1 + x^n)$ for $x \in D$ and let f be defined by

$$\begin{aligned} f(x) &= 0, & 0 \leq x < 1, \\ &= \frac{1}{2}, & x = 1. \end{aligned}$$

If $0 \leq x < 1$, then

$$0 \leq \frac{x^{2n}}{1 + x^n} \leq x^{2n}.$$

By Example 14.8(c), we have $\lim(x^{2n}) = 0$ for $0 \leq x < 1$. Hence it follows from Exercise 15.A that

$$0 \leq \lim(f_n(x)) = \lim(x^{2n}/(1 + x^n)) \leq \lim(x^{2n}) = 0$$

for $0 \leq x < 1$ and it is readily seen from Exercise 15.B that $\lim(f_n(x)) = 0$ for $0 \leq x < 1$. If $x = 1$, then the sequence $(f_n(1)) = (1/2)$ converges to $1/2$. If $x > 1$, then

$$\frac{x^{2n}}{1 + x^n} = \frac{x^{2n}\frac{1}{x^n}}{(1 + x^n)\frac{1}{x^n}} = \frac{x^n}{\left(\frac{1}{x}\right)^n + 1}.$$

The divergence of $x^n/(1 + x^n) \to +\infty$ follows from the fact that $x^n \to +\infty$ for $x > 1$. Therefore, we conclude that (f_n) converges on $[0, 1]$ to f and diverges on $(1, +\infty]$. Since f is not continuous at the point $x = 1$, it follows that the sequence (f_n) does not converges uniformly on D to f. But the sequence (f_n) converges uniformly on $[0, 1]$ or on $[c, +\infty)$, $c > 1$. First, we take $E = [0, 1]$. Note that

$$\begin{aligned}\|f_n - f\|_E &= \sup\{f_n(x) - f(x) : 0 \le x \le 1\} \\ &= \sup\{\frac{x^{2n}}{1+x^n} : 0 \le x \le a\}.\end{aligned}$$

But since $x^{2n}/(1+x^n) \le x^{2n} \le 1^{2n}$, we conclude that $\|f_n - f\|_E = a^{2n}$. Hence (f_n) converges uniformly on the interval $[0, 1]$. Next, we take $F = [c, +\infty)$ with $c > 1$. Note that

$$\begin{aligned}\|f_n - f\|_F &= \sup\{f_n(x) - f(x) : c \le x < +\infty\} \\ &= \sup\{\frac{x^{2n}}{1+x^n} : c \le x < +\infty\}\end{aligned}$$

But since $x^n \ge c^n$, we conclude that $\|f_n - f\|_F = x^{2n}/(1+x^n)$. Hence (f_n) converges uniformly on the interval $[c, +\infty)$ with $c > 1$.

(e) Let $D = \{x \in \mathbf{R} : x \ge 0\}$ and for each natural number n let f_n be defined by $f_n(x) = x^{2n}/(1 + x^{2n})$ for $x \in D$ and let f be defined by

$$\begin{aligned} f(x) &= 0, && 0 \le x < 1, \\ &= \frac{1}{2}, && x = 1, \\ &= 1, && x > 1. \end{aligned}$$

If $0 \le x < 1$, then

$$0 \le \frac{x^{2n}}{1 + x^{2n}} \le x^{2n}.$$

By Example 14.8(c), we have $\lim(x^{2n}) = 0$ for $0 \le x < 1$. Hence it follows from Exercise 15.A that

$$0 \le \lim(f_n(x)) = \lim(x^{2n}/(1 + x^{2n})) \le \lim(x^{2n}) = 0$$

for $0 \le x < 1$ and it is readily seen from Exercise 15.B that $\lim(f_n(x)) = 0$ for $0 \le x < 1$. If $x = 1$, then the sequence $(f_n(1)) = (1/2)$ converges to $1/2$. If $x > 1$, then

$$\frac{x^{2n}}{1 + x^{2n}} = \frac{x^{2n}\frac{1}{x^{2n}}}{(1 + x^{2n})\frac{1}{x^{2n}}} = \frac{1}{\left(\frac{1}{x}\right)^{2n} + 1}.$$

The convergence of $x^{2n}/(1+x^{2n}) \to 1$ follows from the fact that $(1/x)^{2n} \to 0$ for $x > 1$. Therefore, we conclude that (f_n) converges on D to f. However, the sequence (f_n) does not converges uniformly on D to f. Indeed, if $n_k = k$ and $x_k = \left(\frac{1}{2}\right)^{1/2k}$, then

$$|f_k(x_k) - f(x_k)| = \left| \frac{\left[\left(\frac{1}{2}\right)^{1/2k}\right]^{2k}}{1 + \left[\left(\frac{1}{2}\right)^{1/2k}\right]^{2k}} - 0 \right| = \left| \frac{\frac{1}{2}}{1+\frac{1}{2}} \right| = \frac{1}{3},$$

showing that (f_n) does not converge uniformly on D to f. Alternatively, since f is not continuous at the point $x = 1$, it follows that the sequence (f_n) does not converges uniformly on D to f. But the sequence (f_n) converges uniformly on any closed set not containing 1. First, we take $E = [0, a]$ with $a < 1$. Note that

$$\begin{aligned} \|f_n - f\|_E &= \sup\{f_n(x) - f(x) : 0 \le x \le a\} \\ &= \sup\{\frac{x^{2n}}{1+x^{2n}} : 0 \le x \le a\}. \end{aligned}$$

But since $x^{2n}/(1 + x^{2n}) \le x^{2n} \le a^{2n}$, we conclude that $\|f_n - f\|_E = a^{2n}$. Hence (f_n) converges uniformly on the interval $[0, a]$ with $a < 1$. Next, we take $F = [c, +\infty)$ with $b > 1$. Note that

$$\begin{aligned} \|f_n - f\|_F &= \sup\{f_n(x) - f(x) : c \le x < +\infty\} \\ &= \sup\{\frac{x^{2n}}{1+x^{2n}} - 1 : c \le x < +\infty\} \\ &= \sup\{\frac{1}{1+x^{2n}} : c \le x < +\infty. \end{aligned}$$

But since $x^{2n} \ge b^{2n}$, we conclude that $\|f_n - f\|_F = 1/(1+b^{2n})$. Hence (f_n) converges uniformly on the interval $[c, +\infty)$ with $b > 1$.

Exercise (24.J.). Prove the following theorem of G. Polya.[1] If for each $n \in \mathbf{N}$ the function f_n on $\mathbf{I}$ to $\mathbf{R}$ is monotone increasing and if $f(x) = \lim(f(x_n))$ is continuous on $\mathbf{I}$, then the convergence is uniform on $\mathbf{I}$. (Observe that it is not assumed that the f_n are continuous.)

[1]GEORGE POLYA (1887-) was born in Budapest and taught at Zurich and Stanford. He is widely known for his work in complex analysis, probability, number theory, and the theory of inference.

SOLUTION. Let $\epsilon > 0$ be given; since f is uniformly continuous (Theorem 23.3), there is a number $\delta(\epsilon/2)$ such that if x, y belong to $\mathbf{I}$ and $|x-y| < \delta(\epsilon/2)$, then $|f(x) - f(y)| < \epsilon/2$. Divide the domain $\mathbf{I}$ of f into disjoint cells $I_1, \ldots, I_j$ such that if x, y belong to I_k, then $|x - y| < \delta(\epsilon/2)$. Let x_k be any point belonging to the cell I_k, $k = 1, \ldots, j$; by Lemma 17.3, there is a natural number $K(\epsilon/2, x_k)$ such that for all $n \geq K(\epsilon/2, x_k)$ then $|f_n(x_k) - f(x_k)| < \epsilon/2$. Given $x \in \mathbf{I}$, then x belongs to a subinterval $[x_k, x_{k+1}]$ for some integer k in $\{0, 1, \ldots, j-1\}$. Therefore , for such x we have

$$f_n(x) \geq f_n(x_k) > f(x_k) - \frac{\epsilon}{2} > f(x) - \epsilon$$

and

$$f_n(x) \leq f_n(x_{k+1}) < f(x_{k+1}) + \frac{\epsilon}{2} < f(x) + \epsilon$$

so that

$$|f_n(x) - f(x)| < \epsilon.$$

This establish the the uniform convergence of the sequence (f_n) on $\mathbf{I}$.

It follows that f is monotone increaing. Since f is uniformly continuous, if $\epsilon > 0$, let $0 = x_0 < x_1 < \cdots < x_h = 1$ be such that $f(x_j) - f(x_{j-1}) < \epsilon$ and let n_j be such that if $n \geq n_j$; then $|f(x_j) - f_n(x_j)| < \epsilon$. If $n \geq \sup\{n_0, n_1, \ldots, n_h\}$, show that $|f(x) - f_n(x)| < 3\epsilon$ for all $x \in \mathbf{I}$.

EXERCISE (24.N.). If $f_3(x) = x^3$ for $x \in \mathbf{I}$, calculate the nth Bernstein polynomial for f_3. Show directly that this sequence of polyomials converges uniformly to f_3 on $\mathbf{I}$.

SOLUTION. We recall that the Binomial Theorem asserts that

$$(s+t)^n = \sum_{k=0}^{n} \binom{n}{k} s^k t^{n-k},$$

where $\binom{n}{k}$ denotes the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

By direct inspection we observe that

$$\binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{k}{n}\binom{n}{k}.$$

Examining Definition 24.6, formula (24.2) asserts that the $n$the Berstein polynomial for the function $f_3(x) = x^3$ is

$$B_n(x; f_3) = \frac{(n-1)(n-2)}{n^2}x^3 + \frac{3(n-1)}{n^2}x^2 + \frac{1}{n^2}x,$$

which converges uniformly on $\mathbf{I}$ to f_3. Indeed, on multiplying formular (24.6) by $f_3(x)$, we get

$$\begin{aligned} f_3(x) &= \sum_{k=0}^{n} f_3(x)\binom{n}{k}x^k(1-x)^{n-k} \\ &= \sum_{k=0}^{n} x^3\binom{n}{k}x^k(1-x)^{n-k} \end{aligned}$$

Therefore, we obtain the relation

$$\begin{aligned} f_3(x) - B_n(x) &= \sum_{k=0}^{n}\{f_3(x) - f(k/n)\}\binom{n}{k}x^k(1-x)^{n-k} \\ &= \sum_{k=0}^{n}\{x^3 - (k/n)^3\}\binom{n}{k}x^k(1-x)^{n-k} \end{aligned}$$

from which it follows that

$$|f_3(x) - B_n(x)| \leq \sum_{k=0}^{n} |f(x) - f(k/n)| \binom{n}{k}(1-x)^{n-k}.$$

Now f_3 is bounded, say by M, and also uniformly continuous. Let $\epsilon > 0$ and let $\delta(\epsilon)$ be as in the definition of uniform continuity for f_3. It turns out to be convenient to choose n so large that

$$n \geq \sup\{(\delta(\epsilon))^{-4}, M^2/\epsilon^2\},$$

Then

$$\sum_k \epsilon \binom{n}{k} x^k (1-x)^{n-k} \leq \epsilon \sum_{k=1}^{n} \binom{n}{k} x^k (1-x)^{n-k} = \epsilon.$$

The sum taken over those k for which $|x - k/n| \geq n^{-1/4}$, that is, $(x - k/m)^2 \geq n^{-1/2}$, can be estimated. We obtain the upper bound

$$\begin{aligned} \sum_k 2M \binom{n}{k} x^k (1-x)^{n-k} &= 2M \sum_k \frac{(x-k/n)^2}{(x-k/n)^2} \binom{n}{k} x^k (1-x)^{n-k} \\ &\leq 2M\sqrt{n} \left\{ \frac{1}{n} x(1-x) \right\} \leq \frac{M}{2\sqrt{n}}, \end{aligned}$$

since $x(1-x) \leq 1/4$ on the interval $\mathbf{I}$. Recalling the determination (24.11) for n. Hence, for n chosen above we have

$$|f_3(x) - B_n(x)| < 2\epsilon,$$

independently of the value of x. This shows that the sequence (B_n) converges uniformly on $\mathbf{I}$ to f.

Exercise (24.S.). Show that the Weierstrass Approximation Theorem fails for bounded open intervals.

Solution. Let (a, b) be a bounded open interval $(-\infty < a < b < +\infty)$. Let $f(x) = 1/(x-a)$ then f is an unbounded continuous functions on (a, b) which cannot be uniformly approximated by polynomials because polynomials are bounded on bounded intervals. But even you restrict attention to bounded continuous functions on (a, b), the Weierstrass theorem will still fail. Consider $g(x) = sin(1/(x-a))$. If a polynomial uniformly approximates g to within some distance < 1, then the polynomial will have infinitely many zeros, a contradiction.

CHAPTER 25

Limits of Functions

CHAPTER 26

Some Further Results

EXERCISE (26.N.). If $K \subseteq \mathbf{R}^p$ is compact and (f_n) is a sequence of continuous functions on K to $\mathbf{R}^q$ which is uniformly convergent on K, show that the family $\{f_n\}$ is uniformly equicontinuous on K in the sense of Defintition 26.6.

SOLUTION. Let f be the limit on K of the sequence (f_n). Since (f_n) converges uniformly on K to f, given $\epsilon > 0$ there is a natural number $K(\epsilon/3)$ (depending on ϵ but not on $x \in K$) such that for all $n \geq K(\epsilon)$ and $x \in K$, then $\|f_N(x) - f(x)\| < \epsilon/3$. To show that $\{f_n\}$ is uniformly equicontinuous on K in the sense of Defintition 26.6, we note that for each $j < K(\epsilon)$ the function f_j is uniformly continuous so there is a $\delta_j(\epsilon) > 0$ such that if x and y belong to K and $\|x - y\| \leq \delta_j(\epsilon)$, then $\|f_j(x) - f_j(y)\| < \epsilon$. Since f is also uniformly continuous, there is a $\delta'(\epsilon/3) > 0$ such that if x and y belong to K and $\|x - y\| < \delta'(\epsilon/3)$, then $\|f(x) - f(y)\| < \epsilon/3$. We now define δ to be the positive real number

$$\delta = \inf\{\delta_1(\epsilon), \ldots, \delta_{K(\epsilon)-1}(\epsilon), \delta'(\epsilon/3)\}.$$

If x, y belong to K and $\|x - u\| \leq \delta$, then for $n \geq K(\epsilon)$,

$$\begin{aligned}
\|f_n(x) - f_n(u)\| &\leq \|f_n(x) - f(x)\| + \|f(x) - f(y)\| + \|f_n(y) - f(y)\| \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 \\
&= \epsilon.
\end{aligned}$$

Thus this in fact holds for all n since $\delta \leq \delta_j(\epsilon)$ for $j < K(\epsilon)$ as well. This establishes the equicontinuity of the the family $\{f_n\}$ on K.

EXERCISE (26.O.). Let $\mathscr{F}$ be a bounded and uniformly equicontinuous collection of functions with domain on $D \subseteq \mathbf{R}^p$ to $\mathbf{R}$ and let f^* be defined on $D \to \mathbf{R}$ by

$$f^*(x) = \sup\{f(x) : f \in \mathscr{F}\}.$$

Show that f^* is continuous on D to $\mathbf{R}$.

Solution. Since $\mathscr{F}$ is a bounded collection of functions with domain on $D \subseteq \mathbf{R}^p$ to $\mathbf{R}$, so f^* be defined on $D \to \mathbf{R}$ by

$$f^*(x) = \sup\{f(x) : f \in \mathscr{F}\}$$

is well-defined. Since $f^*(x) = \sup\{f(x) : f \in \mathscr{F}\}$, given $\epsilon > 0$ there is a function $f \in \mathscr{F}$ such that $f^*(x) - \epsilon/3 < f(x)$, that is, $|f^*(x) - f(x)| < \epsilon/3$ for all x in D. To show that f^* is continuous at a point a in D, we note that

$$\begin{aligned} |f^*(x) - f^*(a)| &\le |f^*(x) - f(x)| + |f(x) - f(a)| + |f(a) - f^*(a)| \\ &< \epsilon/3 + |f(x) - f(a)| + \epsilon/3. \end{aligned}$$

Since $\mathscr{F}$ is uniformly equicontinuous on D, there is a natural number $\delta = \delta(\epsilon/3, a, f) > 0$ such that if $\|x - a\| < \delta$ and $x \in D$, then $|f(x) - f(y)| < \epsilon/3$. This establishes the continuity of the function f^* at the arbitrary point a in D.

Exercise (26.Q.). Consider the following sequences of functions which show that the Arzela-Ascoli Theorem 26.7 may fail if the various hypotheses are dropped.

(a) $f_n(x) = x + n$ for $x \in [0, 1]$;
(b) $f_n(x) = x^n$ for $x \in [0, 1]$;
(c) $f_n(x) = \frac{1}{1+(x-n)^2}$ for $x \in [0, +\infty)$.

Solution. (a) Domain compact, sequence uniformly equicontinuous but not bounded.

(b) Domain compact, sequence bounded but not uniformly equicontinuous.

(c) Domain not compact, sequence bounded, and uniformly equicontinuous.

Part 5

Functions of One Variable

CHAPTER 27

The Mean Value Theorem

EXERCISE (27.A.). Using the definition, calculate the derivative (when it exists) of the functions given by the expressions
(a) $f(x) = x^2$ for $x \in \mathbf{R}$,
(b) $g(x) = x^n$ for $x \in \mathbf{R}$,
(c) $h(x) = \sqrt{x}$ for $x \geq 0$,
(d) $F(x) = 1/x$ for $x \neq 0$,
(e) $G(x) = |x|$ for $x \in \mathbf{R}$,
(f) $H(x) = 1/x^2$ for $x \neq 0$.

SOLUTION. (b) Write $f(x+h)$ and $f(x)$,

$$\begin{aligned} f(x+h) &= (x+h)^n \\ &= x^n + nx^{n-1} + \frac{n(x-1)}{2}x^{n-2}h^2 + \cdots + h^n \end{aligned}$$

and

$$f(x) = x^n$$

Compute $f(x+h) - f(x)$,

$$\begin{aligned} f(x+h) - f(x) &= x^n + nx^{n-1} + \frac{n(x-1)}{2}x^{n-2}h^2 + \cdots + h^n - x^n \\ &= x^n + nx^{n-1} + \frac{n(x-1)}{2}x^{n-2}h^2 + \cdots + h^n. \end{aligned}$$

Simply $\frac{f(x+h)-f(x)}{h}$

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{nx^{n-1}h + \frac{n(x-1)}{2}x^{n-2}h^2 + \cdots + h^n}{h} \\ &= nx^{n-1} + \frac{n(x-1)}{2}x^{n-2}h + \cdots + h^{n-1} \end{aligned}$$

Compute $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} (nx^{n-1} + \frac{n(x-1)}{2} x^{n-2}h + \cdots + h^{n-1}) \\ &= nx^{n-1} \end{aligned}$$

(f) Write $f(x+h)$ and $f(x)$,

$$\begin{aligned} f(x+h) &= \frac{1}{(x+h)^2}, \\ f(x) &= \frac{1}{x^2}. \end{aligned}$$

Compute $f(x+h)-f(x)$

$$f(x+h)-f(x) = \frac{1}{(x+h)^2} - \frac{1}{x^2} = -\frac{h(h+2x)}{x^2(x+h)^2}.$$

Simply $\frac{f(x+h)-f(x)}{h}$

$$\frac{f(x+h)-f(x)}{h} = -\frac{h(h+2x)}{x^2(x+h)^2h} = -\frac{h+2x}{x^2(x+h)^2}.$$

Compute $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} -\frac{h+2x}{x^2(x+h)^2} \\ &= \frac{2x}{x^4} \\ &= -\frac{1}{x^3}. \end{aligned}$$

Exercise (27.D.). Show that the function defined by

$$g(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is differentiable for all real numbers, by that g' is not continuous at $x = 0$.

SOLUTION. For the first part, suppose $x \neq 0$. Then x^2 and $\sin(1/x)$ are differentiable functions. Thus $x^2 \sin(1/x)$ is also differentiable when $x \neq 0$, by Problem 27.B. Suppose $x = 0$. Consider

$$\begin{aligned}\frac{g(0+h)-g(0)}{h} &= \frac{(0+h)^2 \sin \frac{1}{0+h} - 0}{h} \\ &= \frac{h^2 \sin \frac{1}{h}}{h} \\ &= h \sin \frac{1}{h}\end{aligned}$$

Since $h \sin \frac{1}{h} \leq |h|$(for $\left|\sin \frac{1}{h}\right| \leq 1$) and $|h| \to \infty$ as $h \to \infty$, so

$$\lim_{h\to\infty} h \sin \frac{1}{h} = 0$$

Thus

$$g'(0) = \lim_{h\to\infty} \frac{g(0+h)-g(0)}{h} = \lim_{h\to\infty} h \sin \frac{1}{h} = 0$$

Thus $g(x)$ is differentiable at $x = 0$.

It is easy to show the second part.

EXERCISE (27.E.). The function $h\colon \mathbf{R} \to \mathbf{R}$defined by $h(x) = x^2$for $x \in \mathbf{Q}$ and $h(x) = 0$ for $x \notin \mathbf{Q}$ is continuous at exactly one point. Is it differentiable there?

SOLUTION. For the first part, at $x_0 = 0$, given $\epsilon > 0$, choose $\delta = \sqrt{\epsilon}$. If $|x - 0| < \delta = \sqrt{\epsilon}$. We have$|f(x) - f(0)| = |f(x)| = |x^2| < \epsilon$ if x is rational and $|f(x) - f(0)| = 0 < \epsilon$ if x is irrational. In either case, $|x| < \delta$ implies $|f(x) - f(0)| < \epsilon$, showing that f is continuous at $x = 0$. For $x_0 \neq 0$, we take a sequence $\{y_n\}$ of irrationals converging to x_0. Then $f(y_n) = 0 \to 0 \neq x_0^2 = f(x_0)$.

For the second part, we have

$$\left|\frac{f(0+h)-f(0)}{h}\right| = \left|\frac{f(h)-f(0)}{h}\right| \leq \left|\frac{h^2}{h}\right| = |h|$$

for $f(0) = 0$ and $|h| \leq h^2$. This implies that

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = 0$$

so it is differentiable.

EXERCISE (27.H.). (Darboux) If f is differentiable on $[a, b]$, if $f'(a) = A$, $f'(b) = B$, and if C lies between A and B, then there exists a point c in (a, b) for which $f'(c) = C$. (Hint: consider the lower bound of the function $g(x) = f(x) - C(x - a)$.)

SOLUTION. WLOG, assume that $f'(a) < C < f'(b)$. Let $g(x) = f(x) - C(x - a)$. Then $g'(x) = f'(x) - C$. We have $g'(a) = f'(a) - C < 0$, so that $g(t_1) < g(a)$ for some $t \in (a, b)$, and $g'(b) = f'(b) - C > 0$, so that $g(t_2) < g(b)$ for some $t_2 \in (a, b)$. Hence g attain its minimum on $[a, b]$ at some point x such that $a < x < b$, implying that $g'(x) = 0$. Therefore $f'(x) = C$.

EXERCISE (27.L.). Let $f\colon [a, b] \to \mathbf{R}$ be differentiable at $c \in [a, b]$. Show that if for every $\epsilon > 0$ there is a $\delta(\epsilon)$ such that if $0 < |x - y| < \delta(\epsilon)$, and $a \le x \le c \le y \le b$, then

$$\left|\frac{f(x) - f(y)}{x - y} - f'(c)\right| < \epsilon.$$

SOLUTION. Since f is differentiable at c, it follows that for any ϵ, there is $\delta(\epsilon)$ such that if $|x - c| < \delta$, then

$$\begin{aligned}\left|\frac{f(x) - f(c)}{x - c} - f'(c)\right| &< \frac{\epsilon}{2},\\ \left|\frac{f(y) - f(c)}{y - c} - f'(c)\right| &< \frac{\epsilon}{2}\end{aligned}$$

We have

$$\frac{f(x) - f(y)}{x - y} = \frac{x - c}{x - y} \cdot \frac{f(x) - f(c)}{x - c} - \frac{y - c}{x - y} \cdot \frac{f(y) - f(c)}{x - c}.$$

It implies

$$\frac{f(x) - f(y)}{x - y} - f'(c) = \frac{x - c}{x - y}\left[\frac{f(x) - f(c)}{x - c} - f'(c)\right] - \frac{y - c}{x - y}\left[\frac{f(y) - f(c)}{x - c} - f'(c)\right]$$

Thus

$$\begin{aligned}\left|\frac{f(x) - f(y)}{x - y} - f'(c)\right| &\le \left|\frac{x - c}{x - y}\right|\left|\frac{f(x) - f(c)}{x - c} - f'(c)\right| + \left|\frac{y - c}{x - y}\right|\left|\frac{f(y) - f(c)}{x - c} - f'(c)\right|\\ &\le \left|\frac{f(x) - f(c)}{x - c} - f'(c)\right| + \left|\frac{f(y) - f(c)}{x - c} - f'(c)\right|\\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2}\\ &= \epsilon\end{aligned}$$

EXERCISE (27.P.). A function $f\colon \mathbf{R} \to \mathbf{R}$ is said to be **even** if $f(-x) = f(x)$ for all $x \in \mathbf{R}$, and to be **odd** if $f(-x) = -f(x)$ for all $x \in \mathbf{R}$. If f is differentiable on $\mathbf{R}$ and even (repectively, odd), show that f' is odd (respectively, even).

SOLUTION. Suppose $f(x)$ is even. We shall prove that $f'(-x) = -f'(x)$. Indeed,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{f(-x-h) - f(-x)}{h} \\ &= -\lim_{h\to 0} \frac{f(-x+(-h)) - f(-x)}{-h} \\ &= -f'(-x). \end{aligned}$$

EXERCISE (27.S.). Let $f\colon [0, +\infty) \to \mathbf{R}$ be differentiable on $(0 + \infty)$.
(a) If $f'(x) \to b \in \mathbf{R}$ as $x \to +\infty$, show that for any $h > 0$ we have

$$\lim_{x\to+\infty} \frac{f(x+h) - f(x)}{h} = b.$$

(b) If $f(x) \to a \in \mathbf{R}$ and $f'(x) \to b \in \mathbf{R}$ as $x \to +\infty$, then $b = 0$.
(c) If $f'(x) \to b \in \mathbf{R}$ as $x \to +\infty$, then $f(x)/x \to b$ as $x \to +\infty$.

SOLUTION. (b) Applying part (a) above, for any $h > 0$

$$\begin{aligned} b &= \lim_{x\to+\infty} \frac{f(x+h) - f(x)}{h} \\ &= \frac{\lim_{x\to+\infty}(f(x+h) - f(x))}{h}2 \\ &= \frac{\lim_{x\to+\infty} f(x+h) - \lim_{x\to+\infty} f(x)}{h} \\ &= \frac{a-a}{h} = 0 \end{aligned}$$

(c) Applying the MVT for $[\frac{x}{2}, x]$, we have

$$\begin{aligned} && f(x) - f(\frac{x}{2}) = f'(c)\frac{x}{2} \\ \iff && \frac{f(x)}{x} - \frac{f(\frac{x}{2})}{x} = \frac{1}{2}f'(c). \end{aligned}$$

When $x \to \infty$, then $c \to \infty$, so

$$\lim_{x\to\infty} \frac{f(x)}{x} - \lim_{x\to\infty} \frac{f(\frac{x}{2})}{x} = \frac{1}{2} \lim_{c\to\infty} f'(c) = \frac{1}{2} b. \tag{27.S.1}$$

Let $u = \lim_{x\to\infty} \frac{f(x)}{x}$, then

$$\lim_{x\to\infty} \frac{f(x/2)}{x} = \frac{1}{2} \lim_{x\to\infty} \frac{f(\frac{x}{2})}{\frac{x}{2}} = \frac{1}{2} u \tag{27.S.2}$$

Combine (27.S.1) and (27.S.2), we have $u - \frac{1}{2}u = \frac{1}{2}b$. Thus $u = b$

CHAPTER 28

Further Applications of the Mean Value Theorem

Exercise (28ζ). A function φ on an interval J of $\mathbf{R}$ to $\mathbf{R}$ is said to be (**midpoint**) **convex** in case

$$\varphi(\frac{x_1 + x_2}{2}) \le \frac{1}{2}(\varphi(x_1) + \varphi(x_2))$$

for each x, y in J. (In geometrical terms: the midpoint of any chord of the curve $y = \varphi(x)$, lies above or on the curve.) In this project we shall suppose that φ is a continuous convex function.

(a) If $n = 2^m$ and if $x_1, \ldots, x_n$ belong to J, then

$$\varphi(\frac{x_1 + x_2 + \cdots + x_n}{n}) \le \frac{1}{n}\left[\varphi(x_1) + \varphi(x_2) + \cdots + \varphi(x_n)\right].$$

(b) If $n < 2^m$ and if $x_1, \ldots, x_n$ belong to J, let x_j for $j = n+1, \ldots, 2^m$ be equa; to

$$\overline{x} = (\frac{x_1 + x_2 + \cdots + x_n}{n}).$$

Show that the same inequality holds as in part (a).

(c) Since φ is continuous, show that if x, y belong to J and $t \in I$, then

$$\varphi((1-t)x + ty) \le (1-t)\varphi(x) + t\varphi(y).$$

(In geometrical terms: the entire chord lies above or on the curve.)

(d) Suppose that φ has a second derivative on J. Then a necessary and sufficient condition that φ be convex on J is that $\varphi''(x) \ge 0$ for $x \in J$. (Hint: to prove the necessity, use Exercise 28.O. To prove the sufficiency, use Taylor's Theorem and expand about $\overline{x} = (x+y)/2$.)

(e) If φ is a continuous convex function on J and if $x \le y \le z$ belong to J, show that

$$\frac{\varphi(y) - \varphi(x)}{y - x} \le \frac{\varphi(z) - \varphi(x)}{z - x}.$$

Therefore, if $w \le x \le y \le z$ belong to J, then

$$\frac{\varphi(x) - \varphi(w)}{x - w} \le \frac{\varphi(z) - \varphi(y)}{z - y}.$$

(f) Prove that a continuous convex function φ on J has a left-hand derivative and a right-hand derivative at every point. Furthermore, the subset where φ' does not exist is countable.

SOLUTION. (a) If $n = 2$, then

$$\varphi(\frac{x_1 + x_2}{2}) \le \frac{1}{2}(\varphi(x_1) + \varphi(x_2))$$

Thus the assertion is true for $n = 2$. Assume the assertion is true for $n = 2^k > 2$. That is

$$\varphi(\frac{x_1 + x_2 + \cdots + x_{2^k}}{2^k}) \le \frac{1}{2^k}\left[\varphi(x_1) + \varphi(x_2) + \cdots + \varphi(x_{2^k})\right].$$

Hence if $n = 2^{k+1}$,

$$\begin{aligned}
&\varphi(\frac{x_1 + x_2 + \cdots + x_{2^k} + x_{2^k+1} \cdots + x_{2^{k+1}}}{2^{k+1}}) \\
&= \varphi\left(\frac{\frac{x_1+x_2+\cdots+x_{2^k}}{2^k} + \frac{x_{2^k+1}\cdots+x_{2^{k+1}}}{2^k}}{2}\right) \\
&\le \frac{1}{2}\left[\varphi\left(\frac{x_1 + x_2 + \cdots + x_{2^k}}{2^k}\right) + \varphi\left(\frac{x_{2^k+1} \cdots + x_{2^{k+1}}}{2^k}\right)\right] \\
&\le \frac{1}{2}[\frac{1}{2^k}\left(\varphi(x_1) + \varphi(x_2) + \cdots + \varphi(x_{2^k})\right) \\
&\quad + \frac{1}{2^k}\left(\varphi(x_{2^k+1}) + \varphi(x_{2^k+2}) + \cdots + \varphi(x_{2^{k+1}})\right)] \qquad \text{(By induction hypothesis)} \\
&= \frac{1}{2^{k+1}}[\varphi(x_1) + \varphi(x_2) + \cdots + \varphi(x_{2^k}) + \varphi(x_{2^k+1}) \\
&\quad + \varphi(x_{2^k+2}) + \cdots + \varphi(x_{2^{k+1}})]
\end{aligned}$$

Thus the assertion is true for $n = 2^{k+1}$. Therefore the assertion is true for all $n = 2^m$ by induction principle.

(b) For $2^{m-1} < n < 2^m$

$$\varphi\left(\frac{x_1 + x_2 + \cdots + x_{2^{m-1}} + \cdots + x_n}{n}\right)$$

For $j = n+1, \ldots, 2^m$

$$x_j = \overline{x} = \left(\frac{x_1 + x_2 + \cdots + x_{2^{m-1}} + \cdots + x_n}{n} \right)$$

Thus

$$\begin{aligned} &\varphi\left(\frac{x_1 + x_2 + \cdots + x_{2^{m-1}} + \cdots + x_n + \cdots + x_{2^m}}{2^m} \right) \\ &= \varphi\left(\frac{(x_1 + x_2 + \cdots + x_{2^{m-1}} + \cdots + x_n) + (x_{n+1} + \cdots + x_{2^m})}{2^m} \right), \end{aligned}$$

impliying

$$\begin{aligned} \Longrightarrow \quad & \varphi\left(\frac{n\overline{x} + (2^m - n)\overline{x}}{2^m} \right) \leq \frac{1}{2^m} \sum_{i=1}^{n} \varphi(x_i) + (1 - \frac{n}{2^m})\varphi(\overline{x}) \\ \Longrightarrow \quad & \varphi(\overline{x}) \leq \frac{1}{2^m} \sum_{i=1}^{n} \varphi(x_i) + (1 - \frac{n}{2^m})\varphi(\overline{x}) \\ \Longrightarrow \quad & \left(1 - 1 + \frac{n}{2^m}\right) \varphi(\overline{x}) \leq \frac{1}{2^m} \sum_{i=1}^{n} \varphi(x_i) \\ \Longrightarrow \quad & \frac{n}{2^m} \varphi(\overline{x}) \leq \frac{1}{2^m} \sum_{i=1}^{n} \varphi(x_i) \\ \Longrightarrow \quad & \varphi(\overline{x}) \leq \frac{1}{n} \sum_{i=1}^{n} \varphi(x_i) \\ \Longrightarrow \quad & \varphi(\frac{x_1 + x_2 + \cdots + x_n}{n}) \leq \frac{1}{n} \sum_{i=1}^{n} \varphi(x_i) \end{aligned}$$

(c) If $t = 0$, the result is trivial. So we may assume that $t \in (0, 1]$. Since $\mathbf{Q}$ is dense in $\mathbf{R}$, so there is a sequence $(q_n) \to t$ where $t \in [0, 1]$ for $q_n \in \mathbf{Q}$. Since $q_n \in \mathbf{Q}$,so $q_n = \frac{r_n}{s_n}$ for $r_n, s_n \in \mathbf{N}$.

$$
\begin{aligned}
\varphi\left((1-q_n)x + q_n y\right) &= \varphi\left(\left(1-\frac{r_n}{s_n}\right)x + \frac{r_n}{s_n y}\right) \\
&= \varphi\left(\frac{s_n - r_n}{s_n}x + \frac{r_n}{s_n}y\right) \\
&\leq \frac{\varphi(x) + \cdots + \varphi(x) + \varphi(y) + \cdots + \varphi(y)}{s^n}As \\
&= (s_n - r_n)\,\varphi(x) + r_n(y) \\
&= \left(1-\frac{r_n}{s_n}\right)\varphi(x) + \frac{r_n}{s_n}\varphi(y)
\end{aligned}
$$

Since φ is continuous, so as $n \to \infty$, we obtain $\varphi\left((1-t)x + ty\right) \leq (1-t)\varphi(x) + t\varphi(y)$.

CHAPTER 29

The Riemann-Stieltjes Integral

EXERCISE (29.A.). If f is constant on the interval $[a, b]$, then it is integrable with respect to any function g and

$$\int_a^b f\,dg = f(a)\{g(b) - g(a)\}.$$

EXERCISE (29.B.). If g is as in Example 29.3(c), show that f is integrable with respect to g if and only if f continuous at a.

SOLUTION. Let g be defined on $J = [a, b]$ by

$$g(x) = \begin{cases} 0 & x = a, \\ 1 & a < x \leq b. \end{cases}$$

We shall suppose that f is integrable with respect to g. Let $\epsilon > 0$ and let $P_\epsilon = (x_0, x_1, \ldots, x_n)$ be a parition of $[a, b]$ such that if P, Q are refinements of $P_\epsilon$and $S(P; f, g)$ $S(Q; f, g)$ are any corresponding Riemann-Stieltjes sums, then

$$|S(P; f, g) - S(Q; f, g)| < \epsilon. (*)$$

Let $\delta = x_1 - x_0 > 0$ and $S(Q; f, g) = \sum_i^n f(\xi_i)[g(x_i) - g(x_{i-1})] = f(a)$ (we choose $\xi_1 = a$), Thus

$$S(P; f, g) = f(\xi_1).$$

(we choose $\xi_1 \neq a$). Then $\xi_1 - a < x_1 - x_0 = \delta$ (since P is arefinement of P_ϵ) and (*) becomes $|f(\xi_1) - f(a)| < \epsilon$ so f is continuous at $x = a$.

EXERCISE (29.C.). Let g be defined on $I = [0, 1]$ by $g(x) = 0$ for $0 \leq x \leq \frac{1}{2}$ and $g(x) = 1$ for $\frac{1}{2} < x \leq 1$. Show that f is integrable with repect to g on I if and only if it is continuois at $\frac{1}{2}$ from the right. In this case the value of the integral is $f(\frac{1}{2})$.

SOLUTION. We shall suppose that f is integrable with respect to g. Let $\epsilon > 0$ and let $P_\epsilon = (x_0, x_1, \ldots, x_k, x_{k+1} \ldots, x_n)$ be a parition of $[0, 1]$ such that if P, Q are refinements of $P_\epsilon$and $S(P; f, g)$ $S(Q; f, g)$ are any corresponding Riemann-Stieltjes sums, then

$$|S(P; f, g) - S(Q; f, g)| < \epsilon.(*)$$

Let $\delta = x_{k+1} - x_k > 0$ and

$$S(Q; f, g) = \sum_{i}^{n} f(\xi_{k+1})[g(x_i) - g(x_{i-1})] = f(\frac{1}{2})$$

(we choose $\xi_{k+1} = \frac{1}{2}$), Thus

$$S(P; f, g) = f(\xi_{k+1}).$$

(we choose $\xi_{k+1} \neq \frac{1}{2}$). Then

$$\xi_{k+1} - \frac{1}{2} < x_{k+1} - x_{k-} = \delta$$

(since P is a refinement of P_ϵ) and (*) becomes

$$|f(\xi_{k+1}) - f(\frac{1}{2})| < \epsilon$$

so f is continuous at $x = \frac{1}{2}$.

Conversely, suppose f is continuous from the right at $x = \frac{1}{2}$. Let $\epsilon > 0$ and let $P_\epsilon = (x_0, x_1, \ldots, x_k = \frac{1}{2}, x_{k+1} \ldots, x_n)$ be a parition of $[0, 1]$, we have

$$\begin{aligned} S(P; f, g) &= \sum_{i=1}^{n} f(\xi_i)[g(x_i) - g(x_{i-1})] \\ &= f(\xi_k)(0 - 0) + f(\xi_{k+1})(g(x_{k+1}) - g(\frac{1}{2})) \\ &= f(\xi_{k+1}). \end{aligned}$$

Since f is continuous from the right at $x = \frac{1}{2}$, so for every $\epsilon > 0$ given, there is $\delta(\epsilon) > 0$ such that if $x - \frac{1}{2} < \delta$ then $|f(x) - f(\frac{1}{2})| < \epsilon$. Since $\xi_{k+1} - \frac{1}{2} < \delta$, so $|f(\xi_{k+1}) - f(\frac{1}{2})| < \epsilon$. Hence for any refinement P of P_ϵ then $|S(P; f, g) - f(\frac{1}{2})| < \epsilon$. So f is integrable wih respect to g and

$$\int_a^b f\, dg = f(\frac{1}{2})$$

EXERCISE (29.D.). Show that the function f, given in Exxample 29.3(h) is Riemann integrable on I and that the value of its integral is 0.

SOLUTION. We have

$$f(x) = \begin{cases} 0 & x \notin \mathbf{Q} \\ 1 & x = 0 \\ \frac{1}{n} & x = \frac{m}{n}, (m,n) = 1 \end{cases}$$

Show that the function $f(x)$ is discontinuous at finite point (irrational points). Function is bounded and discontious at finite point is integrable.

EXERCISE (29.E.). If f is integrable on $[a,b]$ with respect to f, then

$$\int_a^b f\,df = \frac{1}{2}\left\{(f(b))^2 - (f(a))^2\right\}$$

(a) Prove this by examining the two Riemann-Stieltjes sums for a partition $P = (x_0, x_1, \ldots, x_n)$ obtained by taking $\xi_k = x_{k-1}$ and $\xi_k = x_k$.

(b) Prove this by using Intagration by Parts Theorem 29.7.

SOLUTION. (a) We shall suppose that f is integrable with respect to f. Let $\epsilon > 0$ and let $P_\epsilon = (x_0, x_1, \ldots, x_k, x_{k+1} \ldots, x_n)$ be a parition of $[a,b]$ such that if $P = (\xi_k \colon \xi_k = x_{k-1}), Q = (\xi_k \colon \xi_k = x_k)$ are refinements of $P_\epsilon$and $S(P;f,f)$ $S(Q;f,f)$ are any corresponding Riemann-Stieltjes sums, then

$$\left|S(P;f,f) - \int_a^b f\,df\right| < \epsilon$$

and

$$\left|S(Q;f,f) - \int_a^b f\,df\right| < \epsilon$$

We have

$$S(P;f,f) = \sum_{k=1}^{n} f(x_{k-1})\{f(x_k) - f(x_{k-1})\}$$

and

$$S(Q;f,f) = \sum_{k=1}^{n} f(x_k)\{f(x_k) - f(x_{k-1})\}$$

Thus

$$
\begin{aligned}
\int_a^b f\,df &= \frac{S(P;f,f)+S(Q;f,f)}{2} \\
&= \frac{1}{2}\left[\sum_{k=1}^{n} f(x_{k-1})\{f(x_k)-f(x_{k-1})\} + \sum_{k=1}^{n} f(x_k)\{f(x_k)-f(x_{k-1})\}\right] \\
&= \frac{1}{2}\left[\sum_{k=1}^{n} \left(f(x_{k-1})f(x_k)-f^2(x_{k-1})\right) + \sum_{k=1}^{n} \left(f^2(x_k)-f(x_k)f(x_{k-1})\right)\right] \\
&= \frac{1}{2}\left[f^2(x_n)-f^2(x_0)\right] \\
&= \frac{1}{2}\left[f^2(b)-f^2(a)\right]
\end{aligned}
$$

(b) Applying Theorem 29.7 with $dg = df$, we have

$$
\begin{aligned}
& \int_a^b f\,df + \int_a^b f\,df = f(b)f(b) - f(a)f(a) \\
\iff \quad & 2\int_a^b f\,df = f^2(b) - f^2(a) \\
\iff \quad & \int_a^b f\,df = \frac{1}{2}\left\{f^2(b)-f^2(a)\right\}
\end{aligned}
$$

Exercise (29.F.). Show directly that if f is the greatest integer function $f(x) = [x]$ defined in Example 29.9(e), then f is not integrable with respect to f on the interval $[0,2]$.

Solution. We have

$$
f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & 1 \le x < 2 \\ 2 & x = 2 \end{cases}
$$

Let $\epsilon > 0$ and let $P_\epsilon = (0 = x_0, x_1, \ldots, x_k = 1, \ldots, x_n = 2)$ be a parition of $[0,2]$. For any P is a refinement of $P_\epsilon$choose $\xi_k = x_{k-1}$ and for any Q is a

refinement of P_ϵ choose $\xi_k = x_k$. We have

$$\begin{aligned}
S(P; f, f) &= \sum_{k=1}^{n} f(x_{k-1})\{f(x_k) - f(x_{k-1})\} \\
&= \sum_{k=1}^{n-1} f(x_{k-1})\{f(x_k) - f(x_{k-1})\} + f(x_{n-1})\{f(x_n) - f(x_{n-1})\} \\
&= 0 + f(x_{n-1})\{f(x_n) - f(x_{n-1})\} = 1\{2-1\} = 1
\end{aligned}$$

and

$$\begin{aligned}
S(Q; f, f) &= \sum_{k=1}^{n} f(x_k)\{f(x_k) - f(x_{k-1})\} \\
&= \sum_{k=1}^{n-1} f(x_k)\{f(x_k) - f(x_{k-1})\} + f(x_n)\{f(x_n) - f(x_{n-1})\} \\
&= 0 + f(x_n)\{f(x_n) - f(x_{n-1})\} = 2\{2-1\} = 2.
\end{aligned}$$

Hence

$$|S(P; f, f) - S(Q; f, f)| = |1 - 2| = 1$$

Thus Cauchy Criteiron is failed for $0 < \epsilon < 1$.

EXERCISE (29.G.). If f is Riemann integrable on $[0, 1]$ then

$$\int_0^1 f\, dx = \lim \left(\frac{1}{n} \sum_{k=1}^{n} f(\frac{k}{n}) \right).$$

EXERCISE (29.H.). Show that if g is not integrable in $[0, 1]$, then the sequence of averages

$$(\frac{1}{n} \sum_{k=1}^{n} g(\frac{k}{n}))$$

may or may not be convergent.

SOLUTION. To show that g is not integrable on $[0, 1]$, and the sequence of avergaes

$$\left(\frac{1}{n} \sum_{k=1}^{n} g(\frac{k}{n}) \right)$$

is convergent. We choose g as follows

$$g(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ 0 & x \notin \mathbf{Q} \end{cases}$$

We notes that $g(x)$ is not Riemann integrable. Let $P_\epsilon = (x_0 = 0, x_1 = \frac{1}{n}, x_2 = \frac{2}{n}, \ldots, x_{n-1} = \frac{n-1}{n}, x_n = 1)$ be a parition of $[0,1]$ such that

$$U(P,g) = \sum_{k=1}^{n} \sup_{\xi_i \in [x_{i-1}, x_i]} f(\xi_i)\{g(x_i) - g(x_{i-1})\} = 1 \qquad \text{when } \xi_i \in \mathbf{Q},$$
$$L(P,g) = \sum_{k=1}^{n} \inf_{\xi_i \in [x_{i-1}, x_i]} f(\xi_i)\{g(x_i) - g(x_{i-1})\} = 0 \qquad \text{when } \xi_i \notin \mathbf{Q}.$$

Thus $|U(P,g) - L(P,g)| = 1$ and the cauchy criterion failed for $0 < \epsilon < 1$. Therefore $g(x)$ defined as above is not integrable. However the sqeuence of averages

$$S_n = \frac{1}{n}\sum_{k=1}^{n} g(\frac{k}{n}) = \frac{1}{n}(1 + 1 + \cdots + 1) = 1$$

converges to 1.

To show that g is not integrable on $[0,1]$, and the sequence of avergaes

$$\left(\frac{1}{n}\sum_{k=1}^{n} g(\frac{k}{n})\right)$$

is not convergent. We choose g (To be integrable g must be bounded, so choose g is not bounded) as follows

$$g(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

We note that $g(x)$ is not Riemann integrable. Since $\lim_{x \to 0^+} f(x) = +\infty$, so for any $M = n^2$ there is $\delta > 0$ such that if $0 < x - 0 < \delta$, then $f(x) > M$. Consider the parition that $|x_1 - x_0| < \delta$ then

$$S_n = \frac{1}{n}\sum_{k=1}^{n} g(\frac{1}{n}) > \frac{M}{n} = \frac{n^2}{n} = n.$$

Therefore the sequence of the averages above can or cannot be convergent.

EXERCISE (29.I.). Show that the function h, defined on I by $h(x) = x$ for x rational and $h(x) = 0$ for x irrational, is not Riemann integrable on I.

SOLUTION. We notes that $g(x)$ is not Riemann integrable. Consider

$$P = (x_0 = 0, x_1 = \frac{1}{n}, x_2 = \frac{2}{n}, \dots, x_{n-1} = \frac{n-1}{n}, x_n = 1)$$

be a parition of $[0,1]$ such that

$$\begin{aligned}
U(P,g) &= \sum_{k=1}^{n} \sup_{\xi_i \in [x_{i-1}, x_i]} h(\xi_i)\{(x_i) - (x_{i-1})\} \\
&= \frac{1}{n} \sum_{i=1}^{n} f(\frac{i}{n}) \\
&= \frac{1}{n^2} \sum_{i=1}^{n} i \\
&= \frac{n(n+1)}{2n^2} \\
&= \frac{n^2+n}{2n^2} \\
&= \frac{1}{2} + \frac{1}{2n} > \frac{1}{2}
\end{aligned}$$

and

$$L(P,g) = \sum_{k=1}^{n} \inf_{\xi_i \in [x_{i-1}, x_i]} h(\xi_i)\{(x_i) - (x_{i-1})\}$$

when $\xi_i \notin \mathbf{Q}$. Therefore $|L(P,g) - U(P,g)| > \frac{1}{2}$ and the Cauchy criterion failed for $0 < \epsilon \leq \frac{1}{2}$. Therefore $g(x)$is not Riemenn integrable.

EXERCISE (29.J.). Suppose that f is Riemann integrable on $[a,b]$. If f_1 is a function on $[a,b]$ to $\mathbf{R}$ such that $f_1(x) = f(x)$ except for a *finite number* of points in [a,b], show that f_1 is Riemann integrable and that

$$\int_a^b f_1 = \int_a^b f.$$

(Thus we can change the value of a Riemann integrable function—or leave it undefined—at a finite number of points.)

EXERCISE (29.K.). Give an example to show that the conclusion of the preceding exercise may fail if the number of exceptional points is infinite.

SOLUTION. Let

$$g_1 = \begin{cases} x & x \in \mathbf{Q} \\ 0 & x \notin \mathbf{Q} \end{cases}$$

and $g(x) = 0$. Then g_1 is not Riemann integrable and g is Riemann integrable.

EXERCISE (29.L.). Let $c \in (a, b)$ and let k be defined on $[a, b]$ by $k(c) = 1$ and $k(x) = 0$ for $c \in [a, b]$, $x \neq c$. If $f \colon [a, b] \to \mathbf{R}$ at c show directly that f is k-integrable, that k is f-integrable, and that

$$\int_a^b f\,dk = \int_a^b k\,df = 0.$$

SOLUTION. We shall prove that f is k-integrable. As above, we consider two cases below.

Case 1: c is not a parition point of P. Since $k(x) = 0, \forall x \neq c$, so

$$\begin{aligned} & S(P; k, g) = k(\zeta_I)[f(c) - f(x_{i_0})] + k(\zeta_{i+1})[f(x_{i_0}) - f(c)] \\ \implies \quad & |S(P, k, g)| \leq |f(c) - f(x_{i_0})| + |f(x_{i_0+1}) - f(c)| < 2\epsilon \end{aligned}$$

(since $|k(x)|\forall x$)

Case 2: c is not a parition point of P. Since $k(x) = 0$, $\forall x \neq c$, so

$$S(P, k, g) = |f(x_{i_0+1}) - f(x_{i_0})| \leq |f(x_{i_0+1}) - f(c)| + |f(c) - f(x_{i_0})| < \epsilon + \epsilon = 2\epsilon$$

So k is f-integrable and $\int_a^b k\,df = 0$.

EXERCISE (29.M.). Suppose that f is g-integrable on $[a, b]$. If $g_1 \colon [a, b] \to \mathbf{R}$ such that $g_1(x) = g(x)$ except for a finite number of points in (a, b) at which f is continuous, then f is g_1-integrable and

$$\int_a^b f\,dg_1 = \int_a^b f\,dg.$$

SOLUTION. Let $S = \{a_1, a_2, \ldots, a_k\}$ be the set of points that $g_1(x) \neq g(x)$. Assume $a_1 < a_2 < \cdots < a_k$. let Δx_j be the open subinterval that $a_j \in \Delta_j$ and $\Delta x_j \cap \Delta x_t = \emptyset, \forall j \neq t$. let $\epsilon > 0$ be given. Since the subinterval

Δx_j ($j = \overline{1,k}$) is finite, so we choose parition such that $\sum_{j=1}^{k} |\Delta x_j| < \epsilon$. Since f is g-integrable on $[a,b]$, so f is g-integrable on

$$B = [a, x_{i_1}] \cup [x_{i_1+1}, x_{i_2}] \cup \cdots \cup [x_{i_k+1}, b]$$

(union of intervals not containing a_j). The parition P_ϵ of f with respect to g on $[a,b]$ is

$$P_\epsilon = (a = x_1 < x_2 < \cdots < x_n = b)$$

where x_{i_j}, $i = \overline{1,n}$ are parition points of P_ϵ. For any refinement P of P_ϵ, we have

$$S(P,f,g) = \sum_{(1)} f(\xi_i)[g(x_{i+1}) - g(x_i)] + \sum_{(2)} f(\xi_i)[g(x_{i+1}) - g(x_i)]$$

where $\sum_{(2)}$ is the sum runs on the refinement of $\cup_{i=1}^{n} \Delta x_j$, $\sum_{(1)}$ is the sum remain. We have

$$S(P,f,g_1) = \sum_{(1)} f(\xi_i')[g_1(x_{i+1}) - g(x_i)] + \sum_{(2)} f(\xi_i')[g_1(x_{i+1}) - g(x_i)]$$

where $\sum_{(1)}$ and $\sum_{(2)}$ are sums taking as above.

$$\begin{aligned}
&|S(P,f,g_1) - S(P,f,g)| \\
&= \left| \sum_{(1)} [f(\xi_i) - f(\xi_i')][g(x_{i+1}) - g(x_i)] + \sum_{(2)} \{f(\xi_i)[g(x_{i+1}) - g(x_i)] - f(\xi_i')[g_1(x_{i+1}) - g_1(x_i)]\} \right| \\
&\leq \left| \sum_{(1)} [f(\xi_i) - f(\xi_i')][g(x_{i+1}) - g(x_i)] \right| + \left| \sum_{(2)} \{f(\xi_i)[g(x_{i+1}) - g(x_i)] - f(\xi_i')[g_1(x_{i+1}) - g_1(x_i)]\} \right| \\
&= \sum_{(1)} |f(\xi_i) - f(\xi_i')|\, |g(x_{i+1}) - g(x_i)| + \sum_{(2)} |\{f(\xi_i)[g(x_{i+1}) - g(x_i)] - f(\xi_i')[g_1(x_{i+1}) - g_1(x_i)]\}|
\end{aligned}$$

Evaluating $\sum_{(1)} |f(\xi_i) - f(\xi_i')|\, |g(x_{i+1}) - g(x_i)|$. On B, we have $g_1(x) = g(x)$(since $S \subset [a,b] \backslash B$) and f is g-integrable on B, so by the Cauchy Criterion then

$$\left| \sum_{(1)} [f(\xi_i) - f(\xi_i')][g(x_{i+1}) - g(x_i)] \right| = |S_B(P,f,g) - S_B(P',f,g)| < \epsilon$$

Since f is g-integrable on B, so taking P_ϵ containing Q_ϵ that is the parition in B. Evaluating

$$
\begin{aligned}
&|\sum_{(2)}\left\{f(\xi_i)[g(x_{i+1})-g(x_i)]-f(\xi_i')[g_1(x_{i+1})-g_1(x_i)]\right\}|\\
&=|\sum_{(2.1)}[f(\xi_i)-f(\xi_i')][g(x_{i+1})-g(x_i)]\\
&\quad+\sum_{(2.2)}\{f(\xi_{m_i+1})[g(a_i)-g(x_{m_i})]+f(\xi_{m_i+1})[g_1(x_{m_{i+2}})-g_1(a_i)]\\
&\quad-f(\xi_{m_i+1}')[g_1(a_i)-g(x_{m_i})]-f(\xi_{m_i+2}')[g(x_{m_i+2})-g_1(a_i)]\}|
\end{aligned}
$$

where $\sum_{(2.1)}$ is the sum runs on the subinterval not cntaining a_i for $i=\overline{1,k}$ and $\sum_{(2.2)}$ is the sum runs on the subitnerval containing a_i for $i=1,k$. Assume $[x_{m_i},x_{m_i}+2]$ is least subinterval consisitng parition point a_i ($i=\overline{1,k}$).

Evaluating

$$
\begin{aligned}
&\sum_{(2.2)}|\{f(\xi_{m_i+1})[g(a_i)-g(x_{m_i})]+f(\xi_{m_i+1})[g_1(x_{m_{i+2}})-g_1(a_i)]-f(\xi_{m_i+1}')[g_1(a_i)-g(x_{m_i})]|\\
&\quad-f(\xi_{m_i+2}')[g(x_{m_i+2})-g_1(a_i)]\}\\
&=\sum_{i=1}^{k}|\{f(\xi_{m_i+1})[g(a_i)-g(x_{m_i})]+f(\xi_{m_i+1})[g_1(x_{m_{i+2}})-g_1(a_i)]\\
&\quad-f(\xi_{m_i+1}')[g_1(a_i)-g(x_{m_i})]-f(\xi_{m_i+2}')[g(x_{m_i+2})-g_1(a_i)]\}|\\
&=\sum_{i=1}^{k}|\{[f(\xi_{m_i+1})-f(\xi_{m_i+2})]g(a_i)+[f(\xi_{m_i+1}')-f(\xi_{m_i+1})]g(x_{m_i})\\
&\quad+[f(\xi_{m_i+2})-f(\xi_{m_i+2}')]g(x_{m_i+2})+[f(\xi_{m_i+2}')-f(\xi_{m_i+1}')]g_1(a_i)\}|\\
&\le 4k\epsilon
\end{aligned}
$$

where

$$\sup\{|g(a_i),|g_1(a_i)|,|f_1(\xi_{m_i+1})|,|f(x_{m_i})|,|g(x_{m_i})|,|g(x_{m_i+2})|\}$$

since when taking $\sum_{(2.1)}[f(\xi_i)-f(\xi_i')][g(x_{i+1})-g(x_i)]$, $|x_{m_i}-a_i|<\delta$, $|x_{m_j+2}-|<\delta$ then $|f(x)-f(a_i)|<\epsilon$ because f is continuos at a_i, $i=\overline{1,k}$)

Evaluating

$$\left|\sum_{(2.1)}[f(\xi_i)-f(\xi_i')][g(x_{i+1})-g(x_i)]\right|.$$

Since f is g-integrable on

$$B'=[x_{i_2},x_{m_1}]\cup[x_{m_1+2},x_{m_2}]\cup\cdots\cup[x_{m_k+2},x_{i_k+1}]$$

so

$$|\sum_{(2.1)}[f(\xi_i)-f(\xi_i')][g(x_{i+1})-g(x_i)]| = |S_{B'}(P,f,g)-S_{B'}(P_i',f,g)| < \epsilon$$

by Cauchy Criterion where $S_{B'}$ is the Riemann-Stieltjes sum of f on B'. Therefore

$$|S(P,f,g_1)-S(P,f,g)| < \epsilon + 4K\epsilon + \epsilon = (2+4K)\epsilon$$

Thus f is g_1-integrable and $\int_a^b f\,dg_1 = \int_a^b f\,dg$.

Exercise (29.N.). Suppose that g is continuous on $[a,b]$, that $x \mapsto g'(x)$ exists and is continuous on $[a,b]\backslash\{c\}$, and that one-sided limits

$$g'(c-) = \lim_{\substack{x \to c \\ x < c}} g'(x), \qquad g'(c+) = \lim_{\substack{x \to c \\ x < c}} g'(x)$$

exists. If f is integrable with respect to g on $[a,b]$, then fg' can be defined at c to be Riemann integrable on $[a,b]$ and such that

$$\int_a^b f\,dg = \int_a^b fg'.$$

(Hint: consider Exercise 27.N.)

Solution. Suppose that $c \in (a,b)$. Let g_1 and g_2 be the restrictions of g to $[a,c]$ and $[c,b]$, respectively. Since g is continous on $[a,b]$, it follows that g_1 is continous on $[a,c]$. On $[a,c]$, since $\lim_{x\to c, x<c} g_1'(x)$, so $g_1'(x)$ exists at $x=c$, and $g_1'(c) = \lim_{x\to c,x<c} g_1'(x)$. We have g_1' is continous on $[a,c]$.Since f is g-integrable on $[a,b]$, and hence on $[a,c]$, it follows that f is g_1-integrable on $[a,c]$ (as $g(x) \neq g_1(x)$ only at c). By Thereom 29.8, we have $\int_a^c f\,dg = \int_a^c f\,dg_1 = \int_a^c fg_1' = \int_a^c fg'$ (as $g'(x) \neq g_1'(x)$ only at c, Probelm 29.J.) Similarly, with g_2 on $[c,b]$, we have $\int_c^b f\,dg = \int_c^b f\,dg_2 = \int_c^b fg_2' = \int_c^b fg'$. Thus $\int_a^b f\,dg = \int_a^c f\,dg + \int_c^b f\,dg = \int_a^c fg' + \int_c^b fg' = \int_a^b fg'$, by Theorem 29.6. (Extension of Thereom 29.8).

Exercise (29.O.). If f is Riemann integrable on $[-5,5]$, show that f is integrable with respect to $g(x)=|x|$ and

$$\int_{-5}^6 f\,dg = \int_0^5 f - \int_{-5}^0 f.$$

SOLUTION. We have

$$g(x) = \begin{cases} x & -5 \le x \le 0 \\ -x & 0 \le x \le 5 \end{cases}$$

Let $g_1(x) = -x$ on $[-5, 0]$ and $g_2(x) = x$ on $[0, +5]$. Then g_1 is differential on $[-5, 0]$ and g_2 is differential on $[0, 5]$. Since f is integrable wrt $h(x) = x$ on $[-5, 5]$, so f is is integrable wrt $h(x) = x$ on $[-5, 0]$. By Thereom 29.8, we have $\int_{-5}^{0} f\,dg = \int_{-5}^{0} f\,dg_1 = -\int_{-5}^{0} f$ (as $g'(x) \ne g_1'(x)$ only at 0, Probelm 29.J.) Similarly, with g_2 on $[c, b]$, we have $\int_0^5 f\,dg = \int_0^5 f\,dg_2 = \int_0^5 f$. Thus

$$\int_{-5}^{5} f\,dg = \int_{-5}^{0} f\,dg + \int_{0}^{5} f\,dg = -\int_{-5}^{0} f + \int_{0}^{5} f,$$

by Theorem 29.6.

EXERCISE (29.P.). If $P = (x_0, x_1, \dots . x_n)$ is a partition of $J = [a, b]$, let $||P||$ be defined to be

$$||P|| = \sup\{x_j - x_{j-1} \colon j = 1, 2, \dots, n\};$$

we call $||P||$ the **norm** of the partition P. Define f to be $(*)$-**integrable** with respect to g on J in case there exists a number A with the property: if $\epsilon > 0$ then there is a $\delta(\epsilon) > 0$ such that if $||P|| < \delta(\epsilon)$ and if $S(P; f, g)$ is any corresponding Riemann-Stieltjes sum, then $|S(P; f, g) - A| < \epsilon$. If this is satisfied the number A is called the $(*)$-**integral** of f with respect to g on J. Show that if f is $(*)$-integrable with respect to g on J, then f is integrable with respect to g (in the sense of Definition 29.2) and that values of these integrals are equal.

SOLUTION. Suppose that f is (*)-integrable on $J = [a, b]$ and $\int_a^b = A$. By he definitoon of the *-integrbale function then for $\epsilon > 0$ given, there is $\delta(\epsilon) < 0$ such that if the parition P satisfying $||P|| < \delta$ then

$$|S(P, f, g) - A| < \epsilon$$

Let P_δ be the parition that $||P_\delta|| = \delta$. For any refinement Q of P_δ, we have $||Q|| < ||P_\delta|| < \delta$ so

$$|S(Q, f, g) - A| < \epsilon$$

Hence f is g-integrbale and $\int_a^b f\,dg = A$.

EXERCISE (29.Q.). Let g be defined on I as in Exercise 29.C. Show that a bounded function f is $(*)$-integrable with respect to g in the sense of the preceding exercise if and only if f is continuous at $\frac{1}{2}$ when the value of the $(*)$-integral is $f(\frac{1}{2})$. If h is defined by

$$\begin{aligned} h(x) &= 0, && 0 \le x < \frac{1}{2}, \\ &= 1, && \frac{1}{2} \le x \le 1, \end{aligned}$$

then h is $(*)$-integrable with respect to g on $[0, \frac{1}{2}]$ and on $[\frac{1}{2}, 1]$ but it is not $(*)$=integrable with respect to g on $[0, 1]$. Hence Theorem 29.6(a) may fail for the $(*)$=integral.

SOLUTION. Given ϵ. For $P = (x_0, x_1, \ldots, x_n)$ is any parition of $[0, \frac{1}{2}]$ and $S(P; h, g)$ is a Riemann-Stieltjes sum corresponding to P. Since

$$S(P; h, g) = \sum_{i=1}^{n} h(\xi_i)\{g(x_i) - g(x_{i-1})\} = 0 < \epsilon$$

so $|S(P; h, g) - 0| = 0 < \epsilon$. It follows that h is (*)-integrable, and $\int_0^{\frac{1}{2}} h\,dg = 0$.

For $Q = (x'_0, x'_1, \ldots, x'_m)$ is any parition of $[\frac{1}{2}, 1]$ and $S(Q; h, g)$ is a Riemann-Stieltjes sum corresponding to P. Since

$$S(Q; h, g) = \sum_{j=1}^{m} h(\xi_j)\{g(x_j) - g(x_{j-1})\} = 1$$

so $|S(P; h, g) - 1| = 0 < \epsilon$. It follows that h is (*)-integrable, and $\int_{\frac{1}{2}}^{1} h\,dg = 1$.

To show that g is not (*)integrable with respect to g on $[0, 1]$, we consider the parition $M = (0 = x_0, x_1, \ldots, x_m = b)$ which does not contain $\frac{1}{2}$ as a parition point and $S(M; h, g)$ is a Riemann-Stieltjes sum corresponding to M. Then choose $\xi_{k+1} \in (x_k, \frac{1}{2})$, we have

$$S(M; h, g) = \sum_{i=1}^{m} h(\xi_i)\{g(x_i) - g(x_{i-1})\} = 0$$

(suppose $\frac{1}{2} \in (x_k, x_{k+1})$) and consider the parition

$$N = (0 = x_0, x_1, \ldots, x_k = \frac{1}{2}, \ldots, x_n = b)$$

and $S(N; h, g)$ is a Riemann-Stieltjes sum corresponding to N. Then

$$S(N; h, g) = \sum_{i=1}^{n} h(\xi_i)\{g(x_i) - g(x_{i-1})\} = 1$$

Thus the Riemann-Stieltjes does not satisfy $|S(M; h, g) - A| < \epsilon$ for any A. Therefore h is not (*)-intagrable wrt g on $[0, 1]$.

Exercise (29.R.). Let $g(x) = x$ for $x \in J$. Show that for this integrator, a function f is integrable in the sense of Definition 29.2 if and only if it is (*)-integrable in the sense of Exercise 29.P.

Exercise. Suppose that $f(x)$ is integrable with respect to $g(x) = x$ on J in the sense of Definition 29.2, then there exists a real number I such that for every $\epsilon > 0$, there is a parition $P_\epsilon = (x_0, x_1, \ldots, x_n)$ of J such that if P is any refinement of P_ϵ and $S(P; f, g)$ is any Riemann-Stieltjes sum corresponding to P, then

$$|S(P; f, g) - I| < \epsilon.$$

Let $\delta(\epsilon) = \inf\{x_i - x_{i-1} \colon i = 1, 2, \ldots n\}$. Then for any Q is a parition of J such that $||Q|| < \delta$, then Q is a refinement of P_ϵ. Then $|S(Q; f, g) - I| < \epsilon$. Thus f is (*)-integrable wrt g.

Converserly, this is a special case of 29.P.

Exercise (29.S.). Let f be Riemann integrable on J and let $f(x) \geq 0$ for $x \in J$. If f is continuous at a point $c \in J$ and if $f(x) > 0$, then

$$\int_a^b f > 0.$$

CHAPTER 30

Existence of the Integral

EXERCISE (30.A.). Show that a bounded function which has at most a finite number of discontinuities is Riemann integrable.

SOLUTION. Let $J = [a, b]$ and f be a bounded function which has n points of discontinuities d_i, $i = 1, 2, \ldots, k$ such that

$$a < d_1 < d_2 < \cdots < d_k < b.$$

Since f is bounded, $|f(x)| \leq B$. Let $[\overline{x}_i, \overline{y}_j]$ be the interval containing d_i. Then

$$J = [a, \overline{x}_1] \cup (\overline{x}_1, \overline{y}_1) \cup [\overline{y}_1, \overline{x}_2] \cup (\overline{x}_2, \overline{y}_2) \cup \cdots \cup (\overline{x}_k, \overline{y}_k) \cup [\overline{y}_k, b]$$

Consider the set

$$J_0 = J \backslash \bigcup_{i=1}^{k} (\overline{x}_i, \overline{y}_i)$$

Then J_0 is closed and bounded, and hence is compact.

The function f is uniformly continous on J_0, given $\epsilon/(b-a) > 0$, there exists a $\delta(\epsilon) > 0$ such that if $x, y \in J_0$ and $|x - y| < \delta(\epsilon)$, then $|f(x) - f(y)| < \epsilon/(b-a)$. Let $P_\epsilon = (z_0, z_1, \ldots, z_r)$ be a parition such that $\overline{x}_i, \overline{y}_i \in P_\epsilon$and $|\overline{x}_i - \overline{y}_i| < \epsilon/Bk$. We have

$$M_j - m_j = \sup\{f(x) - f(y) : x, y \in [x_{j-1}, x_j]\}$$

so

$$M_j - m_j \leq \sup_{x,y \in [x_{j-1}, x_j]} \{|f(x) - f(y)|\} \leq \frac{\epsilon}{b-a}$$

If $P = (x_0, x_1, \ldots, x_n)$ is a refinement of P_ϵ, then also $\sup\{x_j - x_{j-1}\} < \delta(\epsilon)$ and so $M_j - m_j < \epsilon$, whence it follows that

$$\sum_{j=1}^{n}(M_j - m_j)(x_j - x_{j-1}) = \sum_{x_j \neq \overline{y}_i, x_{j-1} \neq \overline{x}_i, \forall i} (M_j - m_j)(x_j - x_{j-1}) + \sum_{j=1}^{k}(M_j - m_j)(\overline{y}_j - \overline{x}_j)$$
$$\leq \frac{\epsilon}{b-a} \sum_{x_j \neq \overline{y}_i, x_{j-1} \neq \overline{x}_i, \forall i} (x_j - x_{j-1}) + B\sum_{j=1}^{k} \frac{\epsilon}{Bk} \leq \frac{\epsilon}{b-a}(b-a) + Bk\frac{\epsilon}{Bk} = 2\epsilon.$$

Thus f is Riemann integrable.

Exercise (30.B.). If $f\colon [a,b] \to \mathbf{R}$ is discontinuous at some point of the interval, then there exists a monotone increasing function g such that f is not g-integrable.

Exercise (30.D.). Give an example of a function f which is not Riemann integrable over J but such that $|f|$ and f^2 are Riemann integrable over J.

Solution. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ -1 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Exercise (30.H.). Suppose that f is integrable with respect to an increasing fnction g on $J = [a,b]$ and let F be defined for $x \in J$ by

$$F(x) = \int_a^x f\,dg.$$

Prove that (a) if g is continuous at c, then F is continuous at c, and (b) if f is positive, then F is increasing.

Exercise (30.J.). If f is Riemann integrable on $J = [a,b]$ and if $F' = f$ on J, then

$$F(b) - F(a) = \int_a^b f.$$

Hint: if $P = (x_0, x_1, \ldots, x_n)$ is a partition of J, write

$$F(b) - F(a) = \sum_{j=1}^{n}\{F(x_j) - F(x_{j-1})\}.$$

Solution. We have

$$
\begin{aligned}
& & F(b)-F(a) &= F'(c)(b-a) \\
\iff & & \sum_{j=1}^{n}\{F(x_j)-F(x_{j-1})\} &= \sum_{j=1}^{n} F'(c_j)(x_j-x_{j-1}) \\
& & &= \sum_{j=1}^{n} f(c_j)(x_j-x_{j-1})
\end{aligned}
$$

where $c_j \in [x_{j-1}, x_j]$. Since f is Riemann integrable, then

$$
\lim_{\max[x_j-x_{j-1}]\to 0} \sum_{j=1}^{n} f(c_j)(x_j-x_{j-1}) = \int_a^b f.
$$

Therefore, $F(b)-F(a) = \int_a^b f$.

Exercise (30.M.). In the First Mean Value Theorem 30.9, assume that p is Riemann integrable (instead of continuous). Show that the conclusion still holds.

Exercise (30.P.). If f is integrable with respect to g on $J=[a,b]$, if φ is continuous and strictly increasing on $[c,d]$, and if $\varphi(c)=a$, $\varphi(d)=b$, then $f\circ\varphi$ is integrable with respect t $g\circ\varphi$ and

$$
\int_a^b f\,dg = \int_c^d (f\circ\varphi)\,d(g\circ\varphi).
$$

Solution. Since the function f is integrable with respect to g over $j=[a,b]$, it follows from Cauchy Criterion for Integrability that for each number $\epsilon>0$ there is a partition Q_ϵ of J such that if P anf Q are refinements of Q_ϵ and if $S(P;f,g)$ and $S(Q;f,g)$ are any correpsonding Riemann-Stieltjes sums, then

$$
|S(P;f,g)-S(Q;f,g)| < \epsilon
$$

Since φ is strictly increasing on $[c,d]$ and $\varphi([c,d])=[a,b]$, so there exists the inverse function $\varphi^{-1}\colon [a,b]\to[c,d]$.

Let $P_\epsilon = \varphi^{-1}(Q_e)$ and $Q' = \varphi^{-1}(Q)$ and $P' = \varphi^{-1}(P)$. Then P_ϵ is a partition of $[c,d]$ and Q' and P' are refinements of P_ϵ. Moreover,

$$\begin{aligned} S(P;f,g) &= \sum_{i=1}^{n} f(\xi_i)\{g(x_i)-g(x_{i-1})\} \\ &= \sum_{i=1}^{n} f\circ\varphi(\eta_i)\{g\circ\varphi(\eta_i)-g\circ\varphi(\eta_{i-1})\} \\ &= S(P', f\circ\varphi, g\circ\varphi) \end{aligned}$$

where $\eta_i = \varphi^1(\xi_i)$. Similarly, $S(Q;f,g) = S(Q', f\circ\varphi, g\circ\varphi)$. Thus

$$|S(P', f\circ\varphi, g\circ\varphi) - S(Q', f\circ\varphi, g\circ\varphi)| = |S(P;f,g) - S(Q;f,g)| < \epsilon$$

Thus $f\circ g$ is Riemann-Stieltjes intergable wrt $g\circ\varphi$ on $[c,d]$.

From preceeding proof, we have

$$S(P;f,g) = S(P', f\circ\varphi, g\circ\varphi)$$

so

$$\int_a^b f\,dg = \int_c^d f\circ\varphi\, d(g\circ\varphi).$$

Exercise (30.Q.). If f is continuous on $[a,b]$ and if

$$\int_a^b fh = 0$$

for all continuous h, then $f(x) = 0$ for all x.

Solution. We proceed by contradiction and suppose that there exists a point $x_0 \in J$ such that $f(x_0) > 0$. Then there exists a positive real number δ such that $f(x) > 0$ for all $x \in (x_0-\delta, x_0+\delta) \subset J$. (Why?)

Since f is continuous at $x_0 \in (a,b)$, it follows that for $\epsilon = f(x_0)/2 > 0$, there exist a positive real number $\delta > 0$ such that if $x_0 - \delta < x < x_0 + \delta$, then $|f(x) - f(x_0)| < \epsilon$. Thus

$$f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon$$

for all $x \in (x_0-\delta, x_0+\delta)$. Hence

$$f(x) > f(x_0) - \epsilon = f(x_0) - f(x_0)/2 = f(x_0)/2 > 0$$

for all $x \in (x_0-\delta, x_0+\delta)$. If we set $\delta_0 = \min(\delta, x_0-a, b-x_0)$, then $f(x) > 0$ for all $x \in (x_0-\delta, x_0+\delta) \subset J$. Let $h = f$, then

$$\int_{x_0-\delta_0}^{x_0+\delta_0} fh = \int_{x_0-\delta_0}^{x_0+\delta_0} f\,dx = f^2(c)(2\delta) > 0$$

where $x_0-\delta < c < x_0+\delta$, by First Mean Value Theorem. By the hypothesis,

$$0 \le \int_{x_0-\delta}^{x_0+\delta} f^2\,dx \le \int_a^b f^2\,dx = 0,$$

so $\int_{x_0-\delta}^{x_0+\delta} f^2 dx = 0$, contrdiction to the fact that $\int_{x_0-\delta}^{x_0+\delta} f^2\,dx > 0$. Thus $f(x) = 0$ for all $x \in (a,b)$. We shall prove that $f(a) = 0$ and $f(b) = 0$. Let (x_n) where $x_n \in (a,b)$ and $x_n \to a$. Since f is continuous at a, so $f(a) = \lim_{n\to\infty} f(x_n) = 0$ ($f(x_n) = 0$ sine $x_n \in (a,b)$). A similar argument for $f(b) = 0$.

CHAPTER 31

Further Properties of the Integral

CHAPTER 32

Improper and Infinite Integrals

CHAPTER 33

Uniform Convergence and Infinite Integrals

Part 6

Infinite Series

CHAPTER 34

Convergence of Infinite Series

EXERCISE (34.A.). Let $\sum a_n$ be a given series and let $\sum b_n$ be one in which the terms are the same as those in $\sum(a_n)$ except that the terms for which $a_n = 0$ have been omitted. Show that $\sum(a_n)$ converges to A if and only if $\sum(b_n)$ converges to A.

SOLUTION. Suppose that $\sum(a_n) \to A$. For every $n \in \mathbf{N}$, there exists a positive integer j such that $t_n = s_{n+j}$. Since the terms got which $a_n = 0$ has been omitted, the sequence of partial sums (t_n) is a subsequence of (s_n). Hence if $\sum(a_n) \to A$, then $\sum(b_n) \to A$.

Conversely, suppose that $\sum(b_n) \to A$. For every $n \in \mathbf{N}$, there exists a positive j such that $s_n = t_{n-j}$ where j is the number of a_i $(i = \overline{1,n})$ such that $a_i = 0$ so $(s_n)$is a subsequence of (t_n). Hence if $\sum(b_n) \to A$, then $\sum(a_n) \to A$.

EXERCISE (34.B.). Show that the convergence of a series is not affected by changing a finite number of its terms. (Of course, the value of the sum may be changed.)

SOLUTION. Let $\sum(a_n)$ be a given series and let $\sum(b_n)$ be one in which the are the same as those in $\sum(a_n)$, except at finite terms for which $a_n \neq b_n$. Let (s_n) and (t_n) be the sequences of paritial sums of $\sum(a_n)$ and $\sum(b_n)$, respectiely. Since $a_n \neq b_n$ at finite terms, so there is a natural number N_0 such that for all $n \geq N_0$, $a_n = b_n$.

Suppose that $\sum(a_n) \to A$. By the Cauchy Criterion, for each $\epsilon > 0$, there is a natural number $M(\epsilon)$ such that for all $m \geq n \geq M(\epsilon)$, $||s_m - s_n|| < \epsilon$. Thus for all $m \geq n \geq \max\{N_0, M(\epsilon)\}$, $||t_m - t_n|| = ||s_m - s_n|| < \epsilon$.

Of course, the value of the sum may be changed. For example, define $\sum(a_n)$ and $\sum(b_n)$ as follows:

$$\sum a_n \text{ where } a_n = 1 \text{ for } n \leq 10 \text{ and } a_n = 0 \text{ for } n > 10$$

and

$$\sum b_n \text{ where } b_n = 0 \text{ for all } n$$

Thus$\sum a_n \to 10$ and $\sum b_n \to 0$.

EXERCISE (34.C.). Show that grouping the terms of a convergent series by introducing parentheses containing a finite number of terms does not destroy the convergence or the value of the limit. However, grouing temrs in a divergent series can produce convergence.

SOLUTION (1). Let $x = \sum(x_n)$, and let $\sum(y_m)$ be a grouping of $\sum(x_n)$. Suppose that we have

$$y_1 = x_1 + \cdots + x_{m_1}, \qquad y_2 = x_{m+1} + \cdots + x_{m_2}, \qquad \cdots.$$

If s_n denotes the nth partial sum of $\sum x_n$ and t_m denotes the mth partial sum of $\sum y_m$, then we have

$$t_1 = y_1 = s_{m_1}, \qquad t_2 = y_1 + y_2 = s_{m_2}, \qquad \cdots.$$

Thus, the sequence (t_m) of partial sums of the grouped series $\sum y_m$ is a subsequence of the sequence (s_n) of partial sums of $\sum x_n$. Since this latter series was assumed to be convergent, so is the grouped series $\sum y_m$.

It is clear that the converse to this theorem is not true. Indeed, the grouping

$$(1-1) + (1-1) + (1-1) + \cdots$$

produces a convergent series from $\sum\limits_{n=0}^{\infty} (-1)^n$, which was seen to be divergent in Exercise 14.Q.(d) since the terms do not approach 0.

SOLUTION (2). Let $x = \sum(x_n)$, and let $\sum(y_k)$ be a grouping of $\sum(x_n)$ by introducing parentheses containing a finite number of terms. Let $\sum(a_n)$ be a given series and let $\sum(b_n)$ be one in which the are the same as those in $\sum(a_n)$, except finite terms are containing in parentheses. Let (s_n) and (t_n) be the sequences of paritial sums of $\sum(a_n)$ and $\sum(b_n)$, respectiely. Since the parentheses containing a finite number of terms, so there is a natural number N_0 such that for all $n \geq N_0$, $t_n = s_n$. Suppose that $\sum(a_n) \to A$. Then for each $\epsilon > 0$, there is a natural number $N(\epsilon)$ such that for all $n \geq M(\epsilon)$, $||s_n - A|| < \epsilon$. Thus for all $n \geq \max\{N_0, M(\epsilon)\}$, $||t_n - A|| = ||s_n - A|| < \epsilon$. Thus $\sum(b_n) \to A$.

EXERCISE (34.D.). Show that if a convergent series of real numbers contain only a finite number of negative terms, then it is absolutely convergent.

Solution. Let $\sum(a_n)$ be a given convergent series of real numbers contain only a finite number of negative terms and let $\sum(b_n)$ be one in which $b_n = a_n$ for $a_n \geq 0$, and $b_n = -a_n$ for $a_n < 0$. $\sum(b_n)$ defined as indicated is the $\sum(|a_n|)$. Let (s_n) and (t_n) be the sequences of paritial sums of $\sum(a_n)$ and $\sum(b_n)$, respectiely. Since $b_n = -a_n$ at a finite number of terms, so there is a natural number N_0 such that for all $m \geq n \geq N_0$, $t_m - t_n = s_m - s_n$.

Suppose that $\sum(a_n) \to A$. By the Cauchy Criterion, for each $\epsilon > 0$, there is a natural number $M(\epsilon)$ such that for all $m \geq n \geq M(\epsilon)$, $|s_m - s_n| < \epsilon$. Thus for all $m \geq n \geq \max\{N_0, M(\epsilon)\}$, $|t_m - t_n| = |s_m - s_n| < \epsilon$. Thus $\sum(b_n)$ converges, that is, $\sum|(a_n)|$ converges.

Exercise (34.E.). Show that if a series of real numbers is conditionanlly convergent, then the series of positive terms is divergent and the series of negative terms is divergent.

Solution. Let $\sum(a_n)$ be a given conditionally convergent series of real numbers. Then $\sum(b_n) = \frac{\sum(a_n)+\sum|a_n|}{2}$ is the series of positive terms and $\sum(c_n) = \frac{\sum(a_n)-\sum|a_n|}{2}$ is the series of negative terms. Since $\sum(a_n)$ is conditionally convergent, so $\sum|a_n| \to \infty$, and hence $\sum(b_n) \to \infty$ and $\sum(c_n) \to \infty$.

Exercise (34.F.). By using partial fractions, show that

$$\text{(a)} \sum_{n=0}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)} = \frac{1}{\alpha}, \qquad \text{if } \alpha > 0,$$

$$\text{(b)} \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}.$$

Solution. (a) Consider the series$\sum(1/[(\alpha+n)(\alpha+n+1)])$ with $\alpha > 0$. By using partial fractions, we can write

$$\frac{1}{(\alpha+k)(\alpha+k+1)} = \frac{1}{\alpha+k} - \frac{1}{\alpha+k+1}.$$

This expression shows that the partial sums are telescoping and hence

$$\begin{aligned}
s_n &= \frac{1}{\alpha(\alpha+1)} + \frac{1}{(\alpha+1)(\alpha+2)} + \cdots + \frac{1}{(\alpha+n)(\alpha+n+1)} \\
&= (\frac{1}{\alpha} - \frac{1}{\alpha+1}) + (\frac{1}{\alpha+1} - \frac{1}{\alpha+2}) + \cdots + (\frac{1}{\alpha+n} - \frac{1}{\alpha+n+1}) \\
&= \frac{1}{\alpha} + (-\frac{1}{\alpha+1} + \frac{1}{\alpha+1}) + (-\frac{1}{\alpha+2} + \frac{1}{\alpha+2}) + \cdots + (-\frac{1}{\alpha+n} + \frac{1}{\alpha+n}) - \frac{1}{\alpha+n+1} \\
&= \frac{1}{\alpha} - \frac{1}{\alpha+n+1}.
\end{aligned}$$

It follows that the sequence (s_n) convergent to $1/\alpha$ if $\alpha > 0$.

(b) Consider the series $\sum(1/[n(n+1)(n+2)])$. By using partial fractions, we can write

$$\frac{1}{k(k+1)(k+2)} = \frac{1}{2k} - \frac{1}{k+1} + \frac{1}{2(k+2)}.$$

This expression shows that the partial sums are telescoping and hence

$$\begin{aligned}
s_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} \\
&= \sum_{k=1}^{n} \left(\frac{1}{2k} - \frac{1}{k+1} + \frac{1}{2(k+2)} \right) \\
&= \sum_{k=1}^{n} \frac{1}{2k} - \sum_{k=1}^{n} \frac{1}{k+1} + \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k+2} \\
&= \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k} + \left(- \sum_{k=1}^{n} \frac{1}{k} + 1 - \frac{1}{n+1} \right)) + \frac{1}{2} \left(\sum_{k=1}^{n} \frac{1}{k} - 1 - \frac{1}{2} + \frac{1}{n+1} + \frac{1}{n+2} \right) \\
&= \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} + 1 - \frac{1}{n+1} + \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2} - \frac{1}{4} + \frac{1}{2(n+1)} + \frac{1}{2(n+2)} \\
&= \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}.
\end{aligned}$$

It follows that the sequence (s_n) convergent to $1/4$.

EXERCISE (34.G.). If $\sum(a_n)$ is a convergent series of real numbers, then is $\sum(a_n^2)$ always convergent? If $a_n \geq 0$, then is it true $\sum(\sqrt{a_n})$ is always convergent?

SOLUTION. We claim that the convergence of $\sum(a_n)$ of real numbers does not imply the convergence of $\sum(a_n^2)$. For if $\sum(a_n)$ is $\sum\left((-1)^n/\sqrt{n}\right)$, this series is convergent, but $\sum 1/n = \sum\left([(-1)^n/\sqrt{n}]^2\right)$ diverges.

We claim that the convergence of $\sum(a_n)$ of real numbers, $a_n \geq 0$ does not imply the convergence of $\sum(\sqrt{n})$. For if $\sum(a_n)$ is $\sum\left(1/n^2\right)$, this series is convergent, but $\sum 1/n = \sum\left(\sqrt{1/n}\right)$ diverges.

EXERCISE (34.H.). If $\sum(a_n)$ is convergent and $a_n \geq 0$, then is $\sum(\sqrt{a_n a_{n+1}})$ convergent?

SOLUTION. Let s_n and t_n be the partial sums of $\sum(a_n)$ and $\sum(\sqrt{a_n a_{n+1}})$, respectiely. Since

$$\begin{array}{rcl} a_1 + a_2 & \geq & 2\sqrt{a_1 a_2} \\ a_2 + a_3 & \geq & 2\sqrt{a_2 a_3} \\ \vdots & \vdots & \vdots \\ a_{n-1} + a_n & \geq & 2\sqrt{a_{n-1} a_n} \end{array}$$

Summing side by side we have

$$2s_n - a_1 - a_n \geq 2t_{n-1}$$

so

$$t_{n-1} \leq s_n - \frac{1}{2}a_1 - \frac{1}{2}a_n \leq s_n$$

for all n. Since (s_n) converges, so (s_n) is bounded, it follows from the above inequality, (t_n) is an increasing sequence and is bounded above. Hence (t_n) converges. Thus $\sum(\sqrt{a_n a_{n+1}})$ is convergent.

EXERCISE (34.I.). Let $\sum(a_n)$ be a series of strictly positive numbers and let b_n, $n \in \mathbf{N}$, be defined to be $b_n = (a_1 + a_2 + \cdots + a_n)/n$. Show that $\sum(b_n)$ always diverges.

SOLUTION. Let t_n and r_n be the partial ums of $\sum(b_n)$ and the harmonic series, respectiely.

$$\begin{aligned}
b_1 &= \frac{a_1}{1} \\
b_2 &= \frac{a_1 + a_2}{2} \\
\vdots &\ \vdots\ \vdots \\
b_n &= \frac{a_1 + a_2 + \cdots + a_n}{n}
\end{aligned}$$

Summing side by side we have

$$\begin{aligned}
t_n &= b_1 + b_2 + \cdots + b_n \\
&= a_1 + (\frac{a_1}{2} + \frac{a_2}{2}) + \cdots + (\frac{a_1}{n} + \frac{a_2}{n} + \cdots + \frac{a_n}{n}) \\
&> a_1(1 + \frac{1}{2} + \cdots + \frac{1}{n}) \\
&= a_1 r_n
\end{aligned}$$

Since $(r_n) \to +\infty$, then t_n diverges.

Exercise (34.J.). Let $\sum(a_n)$ be convergent and let c_n, $n \in \mathbf{N}$, be defined to be weighted means

$$c_n = \frac{a_1 + 2a_2 + \cdots + na_n}{n(n+1)}.$$

Then $\sum(c_n)$ converges and equals $\sum(a_n)$.

Solution. We have

$$\begin{aligned}
b_1 &= \frac{a_1}{2} \\
b_2 &= \frac{a_1 + 2a_2}{2 \cdot 3} = \frac{a_1 + 2a_2}{2} - \frac{a_1 + 2a_2}{3} \\
b_3 &= \frac{a_1 + 2a_2 + 3a_3}{3 \cdot 4} = \frac{a_1 + 2a_2 + 3a_3}{3} - \frac{a_1 + 2a_2 + 3a_3}{4} \\
\vdots &\ \vdots\ \vdots \\
b_n &= \frac{a_1 + 2a_2 + \cdots + na_n}{n(n+1)} = \frac{a_1 + 2a_2 + \cdots + na_n}{n} - \frac{a_1 + 2a_2 + \cdots + na_n}{n+1}
\end{aligned}$$

Hence

$$\begin{aligned} t_n &= b_1 + b_2 + \cdots + b_n \\ &= a_1 + a_2 + \cdots + a_n + \frac{a_1 + 2a_2 + \cdots + na_n}{n+1} \\ &= s_n - u_n \end{aligned}$$

where

$$s_n = \sum_{i=1}^{n} a_i$$

and

$$u_n = \frac{a_1 + 2a_2 + \cdots + na_n}{n+1}$$

We shall prove that if the series $\sum(a_n)$ is convergent, then

$$\lim_{n\to\infty} \frac{a_1 + 2a_2 + \cdots + na_n}{n} = 0.$$

Exercise (34.K.). Let $\sum(a_n)$ be a series of monotone decreasing positive numbers. Prove that $\sum_{n=1}^{\infty}(a_n)$ converges if and only if the series

$$\sum_{n=1}^{\infty} 2^n a_{2^n}$$

converges, This result is often called the **Cauchy Condensation Test**. (Hint: group the terms into blocks as in Examples 34.8(b,d).)

Solution. Let s_n and t_r be the partial sums of $\sum(a_n)$ and $\sum(b_n)$ where $b_n = 2^n a_{2^n}$, respecttively.

$$\begin{aligned} s_n &= a_1 + a_2 + a_3 + a_4 + \cdots + a_n, \\ t_k &= 2a_2 + 4a_4 + 8a_8 + \cdots + 2^k a_{2^k}. \end{aligned}$$

For $n \le 2^k$, we have

$$
\begin{aligned}
s_n &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \cdots + a_n \\
&\le a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (\overbrace{a_{2^k} + \cdots + a_{2^{k+1}-1}}^{2^k \text{ terms}}) \\
&\le a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + \cdots + (a_{2^k} + \cdots + a_{2^k}) \\
&= a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k} \\
&= t_k + a_1
\end{aligned}
$$

Thus

$$
s_n \le t_k + a_1. \tag{34.K.1}
$$

For $n > 2^k$, we have

$$
\begin{aligned}
s_n &\ge a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (\overbrace{a_{2^{k-1}+1} + \cdots + a_{2^k}}^{2^{k-1} \text{ terms}}) \\
&\ge a_1 + a_2 + (a_4 + a_4) + (a_8 + a_8 + a_8 + a_8) + \cdots + (a_{2^k} + \cdots + a_{2^k}) \\
&\ge a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{k-1} a_{2^k} \\
&= \frac{1}{2} t_k + a_1
\end{aligned}
$$

Thus

$$
s_n \ge \frac{1}{2} t_k + a_1. \tag{34.K.2}
$$

From (34.K.1) and (34.K.2), it implies that one series converges if and only if the other series converges.

Exercise (34.L.). Use the Cauchy Condensation Test to discuss the convergence of the p-series $\sum(1/n^p)$.

Solution. We have

$$
a_n = \frac{1}{n^p}, \qquad a_{2^n} = \frac{1}{2^{np}}.
$$

Then

$$
\sum_{n=1}^{\infty} 2^n \frac{1}{2^{np}} = \sum_{n=1}^{\infty} (\frac{2}{2^p})^n = \sum_{n=1}^{\infty} (\frac{1}{2^{p-1}})^n
$$

This series converges if and only i

$$\frac{1}{2^{p-1}} < 1 \iff 2^{p-1} > 1 = 2^0 \iff p - 1 > 0 \iff p > 1.$$

If $0 < p \leq 1$, the series diverges.

EXERCISE (34.M.). Use the Cauchy Condensation Test to show that the series

$$\sum \frac{1}{n \log n} \qquad \sum \frac{1}{n(\log n)(\log \log n)}$$

$$\sum \frac{1}{n(\log n)(\log \log n)(\log \log \log n)}$$

are divergent.

SOLUTION. For the first series, we have

$$a_n = \frac{1}{n \log n}, \qquad a_{2^n} = \frac{1}{2^n \log 2^n} = \frac{1}{2^n n \log 2}.$$

Then

$$\sum 2^n a_{2^n} = \sum \frac{1}{n \log 2}$$

Since $\sum \frac{1}{n}$ diverges, then $\sum \frac{1}{\log 2} \cdot \frac{1}{n}$ also diverges. Therefore the given series diverges

For the second series, we have

$$a_n = \frac{1}{n(\log n)(\log \log n)}, \quad a_{2^n} = \frac{1}{2^n(\log 2^n)(\log \log 2^n)} = \frac{1}{2^n(n \log 2)\log(n \log 2)}.$$

Then

$$\sum 2^n a_{2^n} = \sum \frac{2^n}{2^n(n \log 2)\log(n \log 2)} = \sum \frac{1}{(n \log 2)\log(n \log 2)}$$

Let $m = n \log 2$, then

$$\sum 2^n a_{2^n} = \sum \frac{1}{m \log m}$$

Since $\sum \frac{1}{m \log m}$ diverges, $\sum 2^n a_{2^n}$ also diverges and hence $\sum \frac{1}{n(\log n)(\log \log n)}$ diverges.

For the third series, we have

$$\begin{aligned} a_n &= \frac{1}{n(\log n)(\log\log n)(\log\log\log n)}, \\ a_{2^n} &= \frac{1}{2^n(\log 2^n)(\log\log 2^n)(\log\log\log 2^n)} \\ &= \frac{1}{2^n(n\log 2)\log(n\log 2)\log\log(n\log 2)}. \end{aligned}$$

Then

$$\begin{aligned} \sum 2^n_{2^n} &= \sum \frac{2^n}{2^n(n\log 2)\log(n\log 2)\log\log(n\log 2)} \\ &= \sum \frac{1}{(n\log 2)\log(n\log 2)\log\log(n\log 2)}. \end{aligned}$$

Let $m = n\log 2$. Then

$$\sum 2^n_{2^n} = \sum \frac{1}{m\log(m)\log\log(m)}.$$

Since

$$\sum \frac{1}{m\log(m)\log\log(m)}$$

diverges, $\sum 2^n a_{2^n}$ also diverges and hence

$$\sum \frac{1}{n(\log n)(\log\log n)(\log\log\log n)}$$

diverges.

Exercise (34.N.). Show that if $c > 1$, the series

$$\sum \frac{1}{n(\log n)^c}, \qquad \sum \frac{1}{n(\log n)(\log\log n)^c}$$

are convergent.

Solution. For the first series, we have

$$\begin{aligned} a_n &= \frac{1}{n(\log n)^c}, \\ a_{2^n} &= \frac{1}{2^n(\log 2^n)^c} \\ &= \frac{1}{2^n(n\log 2)^c} = \frac{1}{2^n n^c(\log 2)^c}. \end{aligned}$$

Then

$$\sum 2^n a_{2^n} = \sum \frac{2^n}{2^n n^c (\log 2)^c} = \sum \frac{1}{n^c (\log 2)^c}$$

For $c > 1$, the p-harmonic series $\sum \frac{1}{n^c}$ converges, so $\sum \frac{1}{n^c(\log 2)^c}$ converges. So the given series converges

For the second series, we have

$$\begin{aligned} a_n &= \frac{1}{n(\log n)(\log\log n)^c}, \\ a_{2^n} &= \frac{1}{2^n(\log 2^n)(\log\log 2^n)^c} \\ &= \frac{1}{2^n(n\log 2)(\log(n\log 2))^c}. \end{aligned}$$

Then

$$\sum 2^n a_{2^n} = \sum \frac{2^n}{2^n(n\log 2)(\log(n\log 2))^c} = \sum \frac{1}{(n\log 2)(\log(n\log 2))^c}$$

Let $m = n\log 2$, then

$$\sum 2^n a_{2^n} = \sum \frac{1}{m(\log m)^c}.$$

For $c > 1$, the series $\sum \frac{1}{m(\log m)^c}$ converges so does$\sum 2^n a_{2^n}$. So the given series converges

Exercise (34.O.). Suppose that $(a_n)$is a monotone decreasing sequence of positive numbers. Show that if the series $\sum(a_n)$ converges, then $\lim(na_n) = 0$. Is the converse true?

Solution. For the first part, we have

$$\sum_{k=n+1}^{2n} a_k = a_{n+1} + a_{n+2} + \cdots + a_{2n} \geq na_{2n} = \frac{1}{2}(2n)a_{2n} > 0,$$

$$\sum_{k=n+1}^{2n+1} a_k = a_{n+1} + a_{n+2} + \cdots + a_{2n+1} \geq (n+1)a_{2n+1} = \frac{n+1}{2n+1}(2n+1)a_{2n+1} > 0.$$

But sine $\sum a_n$ converges, so

$$\lim_{n\to\infty} \sum_{n+1}^{2n} a_k = \lim \sum_{k=n+1}^{2n+1} a_k = 0$$

and combine with two facts above, we have

$$\lim_{n\to\infty} na_n = 0$$

The converse false. Consider

$$\sum \frac{1}{n\log n}$$

Even though

$$\lim(na_n) = \lim \frac{n}{n\log n} = \lim \frac{1}{\log n} = 0,$$

but $\sum \frac{1}{n\log n}$ diverges.

EXERCISE (34.P.). If $\lim(a_n) = 0$, then $\sum(a_n)$ and $\sum(a_n + 2a_{n+1})$ are both convergent or both divergent.

SOLUTION. Let s_n and t_n be the partial sums of $\sum(a_n)$ and $\sum(b_n)$ where $b_n = a_n + 2a_{n+1}$, respectively. Then we have

$$s_n = \sum_{k=1}^{n} a_k$$

$$\begin{aligned} t_n &= \sum_{k=1}^{n} b_k = \sum_{k=1}^{n}(a_k + 2a_{k+1}) = \sum_{k=1}^{n}(a_k) + \sum_{k=1}^{n}(2a_{k+1}) \\ &= s_n + 2\sum_{k=1}^{n}(2a_k) - 2a_1 + 2a_{n+1} \\ &= s_n + 2s_n - 2a_1 + 2a_{n+1} = 3s_n - 2a_1 + 2a_{n+1} \end{aligned}$$

Since $\lim_{n\to\infty} = 0$, so there exists $M > 0$ such that $|a_n| \leq M$, $\forall n$. Thus

$$t_n \leq 3s_n - 2_1 + 2M. \tag{34.P.1}$$

Since $t_n = 3s_n - 2a_1 + 2a_{n+1}$, so $s_n = \frac{1}{3}t_n + \frac{2}{3}a_1 - \frac{2}{3}a_{n+1}$. Since $\lim_{n\to\infty} a_n = 0$, so for $\epsilon = 1 > 0$ there exists $n_0 \in N$ such that if $n > m_0$ then $-1 < a_n < 1$. Hence

$$s_n = \frac{1}{3}t_n + \frac{2}{3}a_1 - \frac{2}{3}a_{n+1} < \frac{1}{3}t_n + \frac{2}{3}a_1 + \frac{2}{3} \tag{34.P.2}$$

for avery $n > m_0$. From (34.P.1) and (34.P.2), we implies $\sum a_n$ and $\sum(a_n + 2a_{n+1})$ are both convergent or both divergent.

CHAPTER 35

Tests for Absolute Convergence

Exercise (35.A.). Establish the convergence or the divergence of the series whose nth term is given by

(a) $\frac{1}{(n+1)(n+2)}$,
(b) $\frac{n}{(n+1)(n+2)}$,
(c) $2^{-1/n}$,
(d) $n/2^n$,
(e) $[n(n+1)]^{-1/2}$,
(f) $[n^2(n+1)]^{-1/2}$,
(g) $n!/n^n$,
(h) $(-1)^n n/(n+1)$.

Exercise (35.D.). Discuss the convergence or the divergence of the series with nth term (for sufficiently large n) given by

(a) $[\log n]^{-p}$,
(b) $[\log n]^{-n}$,
(c) $[\log n]^{-\log n}$,
(d) $[\log n]^{-\log\log n}$,
(e) $[n \log n]^{-1}$,
(f) $[n(\log n)(\log\log n)^2]^{-1}$.

Exercise (35.E.). Show that the series

$$\frac{1}{1^2} + \frac{1}{2^3} + \frac{1}{3^2} + \frac{1}{4^3} + \cdots$$

is convergent, but that bith the Ratio and the Root Tests fail to apply.

Solution. Let a_n be the nth term of the given series

$$a_n = \begin{cases} \frac{1}{n^2} & \text{if } n \text{ is odd} \\ \frac{1}{n^3} & \text{if } n \text{ is even} \end{cases}$$

so $a_n \leq \frac{1}{n^2}$ for all n. But the series $\sum \frac{1}{n^2}$ converges, so the series $\sum a_n$ given is also convergent by the Comparision Test.

For the Ratio Test, if n is odd then $a_n = \frac{1}{n^2}$, then

$$a_{n+1} = \frac{1}{(n+1)^3}$$

and

$$\frac{\frac{1}{(n+1)^3}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^3} \to 0$$

If n is even then $a_n = \frac{1}{n^3}$ and

$$a_{n+1} = \frac{1}{(n+1)^2}$$

and

$$\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^3}} = \frac{n^3}{(n+1)^2} \to +\infty$$

Thus $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ does not exist.

For the Root Test, we apply the fact that

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1.$$

If n is odd then $a_n = \frac{1}{n^2}$ and

$$(\frac{1}{n^2})^{\frac{1}{n}} = \frac{1}{(n^{\frac{1}{n}})^2} \to 1$$

as $n \to \infty$. If n is even then $a_n = \frac{1}{n^3}$ and

$$(\frac{1}{n^3})^{\frac{1}{n}} = \frac{1}{(n^{\frac{1}{n}})^3} \to 1$$

as $n \to \infty$. So we cannot apply the Root test

Exercise (35.F.). If a and b are positive numbers, then

$$\sum \frac{1}{(an+b)^p}$$

converges if $p > 1$ and diverges if $p \leq 1$.

SOLUTION. Consider

$$(x_n) = \left(\frac{1}{(an+b)^p}\right)$$

and

$$(y_n) = \left(\frac{1}{n^p}\right).$$

We have

$$\frac{x_n}{y_n} = \frac{\frac{1}{(an+b)^p}}{\frac{1}{n^p}} = \left(\frac{n}{an+b}\right)^p,$$

so

$$\lim\left(\frac{x_n}{y_n}\right) = \frac{1}{a^p} \neq 0.$$

Hence $\sum(x_n)$ is convergent iff$\sum(y_n)$ is convergent. $\sum(y_n)$ is convergent iff $p > 1$. Thus $\sum(x_n)$ is convergent iff $p > 1$.

EXERCISE (35.G.). Discuss the series whose nth term is
(a) $\frac{n!}{3\cdot5\cdot7\cdots(2n+1)}$,
(b) $\frac{(n!)^2}{(2n)!}$,
(c) $\frac{2\cdot4\cdots(2n)}{3\cdot5\cdots(2n+1)}$,
(d) $\frac{2\cdot4\cdots(2n)}{5\cdot7\cdots(2n+3)}$.

SOLUTION. (a) Ratio test:

$$\frac{\frac{(n+1)!}{3\cdot5\cdot7\cdots(2n+1)\cdot(2n+3)}}{\frac{n!}{3\cdot5\cdot7\cdots(2n+1)}} = \frac{n+1}{2n+3} \to \frac{1}{2} < 1$$

as $n \to \infty$.
(b) Ratio test:

$$\frac{\frac{[(n+1)]^2}{[2(n+1)]!}}{\frac{(n!)^2}{(2n)!}} = \frac{[(n+1)!]^2}{2(n!)^2(n+1)(2n+1)} = \frac{(n+1)^2}{2(n+1)(2n+1)} \to \frac{1}{4}$$

as $n \to \infty$.
(c) Ratio test:

$$\frac{\frac{2\cdot4\cdots(2n)(2n+2)}{3\cdot5\cdots(2n+1)\cdot(2n+3)}}{\frac{2\cdot4\cdots(2n)}{3\cdot5\cdots(2n+1)}} = \frac{2\cdot4\cdots(2n)\cdot(2n+2)}{3\cdot5\cdots(2n+1)\cdot(2n+3)}\frac{3\cdot5\cdots(2n+1)}{2\cdot4\cdots(2n)} = \frac{2n+2}{2n+3} = 1-\frac{1}{2n+3} \geq 1-\frac{1}{2n}$$

(d) Ratio test:

$$\frac{\frac{2\cdot4\cdots(2n)\cdot(2n+2)}{5\cdot7\cdots(2n+3)\cdot(2n+5)}}{\frac{2\cdot4\cdots(2n)}{5\cdot7\cdots(2n+3)}} = \frac{2\cdot4\cdots(2n)\cdot(2n+2)}{5\cdot7\cdots(2n+3)\cdot(2n+5)}\frac{5\cdot7\cdots(2n+3)}{2\cdot4\cdots(2n)} = \frac{2n+2}{2n+5} = 1-\frac{3}{2n+5} \leq 1-\frac{4}{3n}$$

for all $n \geq 20$.

Exercise (35.H.). The series given by

$$(\frac{1}{2})^p + (\frac{1\cdot3}{2\cdot4})^p + (\frac{1\cdot3\cdot5}{2\cdot4\cdot6})^p + \cdots$$

converges for $p > 2$ and diverges for $p \leq 2$.

Solution. We have

$$\frac{\left(\frac{1\cdot3\cdot5\cdots(2n-1)\cdot(2n+1)}{2\cdot4\cdot6\cdots(2n)(2n+2)}\right)^p}{\left(\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}\right)^p} = \left(\frac{1\cdot3\cdot5\cdots(2n-1)\cdot(2n+1)}{2\cdot4\cdot6\cdots(2n)(2n+2)}\frac{2\cdot4\cdot6\cdots(2n)}{1\cdot3\cdot5\cdots(2n-1)}\right)^p$$

$$= \left(\frac{2n+1}{2n+2}\right)^p = \left(1-\frac{1}{2(n+1)}\right)^p.$$

Using Bernoulli's inequality

$$\left(1-\frac{1}{2(n+1)}\right)^p \geq 1+p\left[-\frac{1}{2(n+1)}\right] = 1-p\frac{1}{2(n+1)} = 1-\frac{p}{2(n+1)} > 1-\frac{p}{2n}$$

In case $p \leq 2$, the given series diverge by Raabe' test.

Consider $p > 2$, we want to prove

$$\left(1-\frac{1}{2(n+1)}\right)^p \leq 1-\frac{p}{2n}.$$

Consider

$$f(x) = (1-x)^p - 1 + \frac{px}{1-2x}$$

for $0 < x < 1$, and so

$$f'(x) = -p(1-x)^{p-1} + \frac{p(1-2x)+2px}{(1-2x)^2} = -p(1-x)^{p-1} + \frac{p}{(1-2x)^2} = \frac{-(1-x)^{p+1}+1}{(1-2x)^2}\cdot p > 0$$

$\forall x \in (0,1)$ for $p > 2$. For $0 < x < 1$,

$$f(x) < f(1) = -1 - p < 0$$

for $p > 2$. Thus

$$(1-x)^p - 1 + \frac{px}{1-2x} < 0$$

for $0 < x < 1$. Hence

$$(1-x)^p < 1 - \frac{px}{1-2x}$$

for $0 < x < 1$. Let $x = \frac{1}{2n+1}$, we have

$$(1 - \frac{1}{2n+2})^p < 1 - \frac{p\frac{1}{2n+1}}{1 - \frac{2}{2n+2}} = 1 - \frac{p}{2n}$$

By Raabe' test, for $p > 2$, the given series convergent.

Exercise (35.N.). If $p > 0$, $q > 0$, then the series

$$\sum \frac{(p+1)(p+2)\cdots(p+n)}{(q+1)(q+2)\cdots(q+n)}$$

converges for $q > p+1$ and diverges for $q \le p+1$.

Exercise (35.O.). Show that the series $\sum(2^n n!)2/(2n+1)!$ is divergent.

Solution. We have

$$\begin{aligned}
\frac{\frac{\left[2^{n+1}(n+1)!\right]^2}{(2n+3)!}}{\frac{(2^n n!)^2}{(2n+1)!}} &= \frac{2^{2n+2}\left[(n+1)!\right]^2}{(2n+3)!}\frac{(2n+1)!}{2^{2n}\left(n!\right)^2} \\
&= \frac{2^2(n+1)^2}{(2n+2)(2n+3)} \\
&= \frac{(2n+2)^2}{(2n+2)(2n+3)} \\
&= \frac{2n+2}{2n+3} \\
&= 1 - \frac{1}{2n+3} \\
&\geq 1 - \frac{1}{2n}.
\end{aligned}$$

By Raabe's test, it is divergent.

CHAPTER 36

Further Results for Series

CHAPTER 37

Series of Functions

Exercise (37.A.). Discuss the convergence and uniform convergence of the series $\sum(f_n)$, where $f_n(x)$ is given by
(a) $(x^2+n^2)^{-1}$,
(b) $(nx)^{-2}$, $x \neq 0$,
(c) $\sin(x/n^2)$,
(d) $(x^n+1)^{-1}$, $x \geq 0$,
(e) $x^n(x^n+1)^{-1}$, $x \geq 0$,
(f) $(-1)^n(n+x)^{-1}$, $x \geq 0$.

Solution. (a) Consider the series $\sum_{n=1}^{\infty}(x^2+n^2)^{-1}$. For all x in $\mathbf{R}$, $\left|(x^2+n^2)^{-1}\right| \leq 1/n^2$. Since the series $\sum(1/n^2)$ is convergent, it follows from the Weierstrass M-test that the given series is uniformly convergent for all x in $\mathbf{R}$.

(e) If $x=0$, then $f_n(0)=0^n(0^n+1)^{-1}=0$, so the series $\sum x^n(x^n+1)$ is absolutely convergent at $x=0$.

If $x=1$, then $\lim_{n\to+\infty} x^n(x^n+1)^{-1}=1/2 \neq 0$, so the series $\sum x^n(x^n+1)^{-1}$ is divergent at $x=1$.

If $0<x<1$, then a direct appliation of the Abel's Test (with $\varphi_n = (x^n+1)^{-1}$and $f_n=x^n$) shows that $\sum x^n(x^n+1)^{-1}$ is uniformly convergent on the interval $(0,1)$.

If $x>1$, then $\lim_{n\to+\infty} x^n(x^n+1)^{-1}=1 \neq 0$, so the series $\sum x^n(x^n+1)^{-1}$ is divergent on the interval $x=(1,+\infty)$.

Exercise (37.B.). If $\sum(a_n)$ is an absolutely convergent series, then the series $\sum(a_n \sin nx)$ is absolutely and uniform convergent.

Solution. A direct application of the M-test (with $M_n=a_n$) shows that $\sum(a_n \sin nx)$ is uniformly convergent for all x in $\mathbf{R}$.

Exercise (37.F.). Dicuss the case $R=0$, $R=+\infty$ in the Cauchy-Hadamard Theorem 37.13.

SOLUTION. We shall treat the case where $R = 0$ when $\rho = \limsup \left(|a_n|^{1/n} \right) = +\infty$, that is, the sequence $\left(|a_n|^{1/n} \right)$ is not bounded. If $|x| \neq 0$, then $|x| > R = 0$. Since $1/|x| < \limsup \left(|a_n|^{1/n} \right) = +\infty$, therefore, there are infinitely many $n \in \mathbf{N}$ for which we have $|a_n|^{1/n} > 1/|x|$. Therefore, $|a_n x^n| > 1$ for infinitely many n, so that the sequence $(a_n x^n)$ does not converges to zero. Thus $\sum (a_n x^n)$ is not absolutely convergent.

If $|x| = 0$, then $|a_n x^n| = 0$ for all n. Thus the series $\sum (a_n x^n)$ is absolutely convergenct.

EXERCISE (37.G.). Show that the redius of convergence R of the power series $\sum (a_n x^n)$ is given by $\lim(|a_n| / |a_{n+1}|)$ whenever this limit exists. Give an example of a power series where this limit does not exist.

SOLUTION. Consider the series

$$1x - \frac{1}{2}x^2 - \frac{1}{4}x^3 + \frac{1}{3}x^4 - \frac{1}{6}x^5 - \frac{1}{8}x^6 + \frac{1}{5}x^7 - \cdots - \frac{1}{4n-6}x^{3n-4} - \frac{1}{4(n-1)}x^{3n-3} + \frac{1}{2n-1}x^{3n-2} - \cdots .$$

We have

$$\begin{aligned} a_{3n-4} &= -\frac{1}{4n-6}, \\ a_{3n-3} &= -\frac{1}{4(n-1)}, \\ a_{3n-2} &= \frac{1}{2n-1}. \end{aligned}$$

Note that

$$\lim \frac{|a_{3n-4}|}{|a_{3n-3}|} = \lim \frac{\left| -\frac{1}{4n-6} \right|}{\left| -\frac{1}{4(n-1)} \right|} = 1$$

but

$$\lim \frac{|a_{3n-3}|}{|a_{3n-2}|} = \lim \frac{\left| -\frac{1}{4(n-1)} \right|}{\left| \frac{1}{2n-1} \right|} = \frac{1}{2}.$$

Thus $\lim(|a_n| / |a_{n+1}|)$ does not exist.

However this series converges. For example, if $x = 1$, let t_n be the partial sum of the series correpsonding to $x = 1$.

$$\begin{aligned}
T_{3n} &= 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \cdots - \frac{1}{4n-6} - \frac{1}{4(n-1)} + \frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n} \\
&= (1 - \frac{1}{2}) - \frac{1}{4} + (\frac{1}{3} - \frac{1}{6}) + \cdots + (\frac{1}{2n-1} - \frac{1}{4n-2}) - \frac{1}{4n} \\
&= \frac{1}{2}(1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{2n}) = \frac{1}{2}S_n,
\end{aligned}$$

where S_n is the partual sum of the series $\sum(-1)^{n+\frac{1}{n}}$, so

$$\lim_{n\to+\infty} T_{3n} = \frac{1}{2}\lim_{n\to\infty} S_n = \frac{1}{2}\ln 2.$$

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Exercise (37.H.). Determine the radius of convergence of the series $\sum(a_n x^n)$, where a_n is given by
(a) $1/n^n$,
(b) $n^\alpha/n!$,
(c) $n^n/n!$,
(d) $(\log n)^{-1}$, $n \geq 2$.
(e) $(n!)^2/(2n)!$,
(f) $n^{-\sqrt{n}}$.

Solution. (a) Since $\lim_{n\to+\infty} \sqrt{\left|\frac{1}{n^n}\right|} = \lim_{n\to+\infty} \frac{1}{n} = 0$, so the power series has radius of convergence equal to $+\infty$.

(b) Since

$$\lim_{n\to+\infty} \frac{n^{\alpha+1}}{(n+1)!}\frac{n!}{n^\alpha} = \lim_{n\to+\infty} \frac{n}{n+1} = 1,$$

so the power series has radius of convergence equal to 1.

CHAPTER 38

Fourier Series

Part 7

Differentiation in $\mathbf{R}^p$

CHAPTER 39

The Derivative in $\mathbf{R}^p$

EXERCISE (39.A.). Let $f\colon \mathbf{R} \to \mathbf{R}$ be defined by

$$\begin{aligned} f(x,y) &= \frac{x}{y} && \text{for } y \neq 0, \\ &= 0 && \text{for } y = 0. \end{aligned}$$

Show that the partial derivatives $D_1 f(0,0)$, $D_2 f(0,0)$ exist and equal 0. However, the derivative of f at $(0,0)$ with respect to a vector $u = (a,b)$ does not exist if $ab \neq 0$. Show that f is not continuous at $(0,0)$; indeed, f is not even bounded on a neighborhood of $(0,0)$.

SOLUTION. We consider the sequence

$$u_n = ((x_n, y_n)) = ((\frac{1}{n}, \frac{1}{n^2})).$$

.As $n \to \infty$, $u_n = (x_n, y_n) \to (0,0)$. We have

$$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n^2}} = \lim_{n\to\infty} n = +\infty.$$

Thus the function $f(x,y)$ is not bounded on a neighborhood of $(0,0)$.

EXERCISE (39.B.). Let $g\colon \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$\begin{aligned} g(x,y) &= 0 && \text{for } xy = 0, \\ &= 1 && \text{for } xy \neq 0. \end{aligned}$$

Show that the partial derivatives $D_1 f(0,0)$, $D_2 f(0,0)$ exist and equal 0. However, the derivative of g at $(0,0)$ with respect to a vector $u = (a,b)$ does not exist if $ab \neq 0$. Show that g is not continuous at $(0,0)$; however, g is bounded on a neighborhood of $(0,0)$.

SOLUTION. We have
$$D_1g(0,0) = \lim_{h\to 0} \frac{g(0+h,0)-g(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$
and
$$D_2g(0,0) = \lim_{k\to 0} \frac{g(0,0+k)-g(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0.$$
Consider the sequences $u_n = ((\frac{1}{n}, \frac{1}{n}))$ and $v_n = ((\frac{1}{n}, 0))$. As $n \to \infty$, u_n and v_n both converge to $(0,0)$. But
$$\lim_{n\to\infty} g(u_n) = \lim_{n\to\infty} g((\frac{1}{n}, \frac{1}{n})) = 1$$
and
$$\lim_{n\to\infty} g(u = v_n) = \lim_{n\to\infty} g((\frac{1}{n}, 0)) = 0.$$
Thus $\lim_{n\to\infty} g((x_n, y_n))$ does not exists as $(x_n, y_n) \to (0,0)$ so $g(x,y)$ is not continuous at $(0,0)$.

We shall prove $g(x,y)$ is bounded on a neighborhood of $(0,0)$. Consider
$$V = B((0,0),1) = \{(x,y) \in \mathbf{R}^2 \colon x^2 + y^2 < 1\},$$
so V is a neighborhood of $(0,0)$ and we have $|g(x,y)| \leq 1, \forall (x,y) \in V$. Thus $g(x,y)$ is bounded on V.

EXERCISE (39.C.). Let $k \colon \mathbf{R}^2 \to \mathbf{R}$ be defined by
$$h(x,y) = \begin{cases} 0 & \text{for } (x,y) = (0,0), \\ \frac{xy}{x^2+y^2} & \text{for } (x,y) \neq (0,0). \end{cases}$$
Show that the partial derivative $D_1h(0,0)$, $D_2h(0,0)$ exist and equal 0. However, the derivative of h at $(0,0)$ with respect to a vetor $u = (a,b)$ does not exist if $ab \neq 0$. Show that h is not continuos at $(0,0)$.

SOLUTION. We have
$$D_1h(0,0) = \lim_{\Delta x\to 0} \frac{h(\Delta x,0)-h(0,0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\frac{\Delta x(0)}{(\Delta x)^2+0^2} - 0}{\Delta x} = 0$$
and
$$D_2h(0,0) = \lim_{\Delta y\to 0} \frac{h(0,\Delta y)-h(0,0)}{\Delta y} = \lim_{\Delta y\to 0} \frac{\frac{(0)\Delta y}{0^2+(\Delta y)^2} - 0}{\Delta x} = 0.$$

Since

$$\begin{aligned}
\lim_{t\to 0}\frac{h(ta,tb)-h(0,0)}{t} &= \lim_{t\to 0}\frac{\frac{(ta)(tb)}{(ta)^2+(tb)^2}-0}{t}\\
&= \lim_{t\to 0}\frac{t^2ab}{t^3(a^2+b^2)}\\
&= \lim_{t\to 0}\frac{ab}{t(a^2+b^2)} = \infty.
\end{aligned}$$

so $D_uh(0,0)$ does not exist.

EXERCISE (39.D.). Let $k\colon \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$k(x,y) = \begin{cases} 0 & \text{for } (x,y) = (0,0),\\ \frac{xy^2}{x^2+y^4} & \text{for } (x,y) \neq (0,0). \end{cases}$$

Show that the partial derivative of k at $(0,0)$ with respect to any vector $u = (a,b)$ exists and that

$$D_uh(0,0) = \frac{b^2}{a} \qquad \text{if } a \neq 0.$$

Show that k is not continuous and hence not differentiable at $(0,0)$.

SOLUTION. Since

$$\lim_{t\to 0}\frac{h(ta,tb)-h(0,0)}{t} = \lim_{t\to 0}\frac{\frac{(ta)(tb)^2}{(ta)^2+(tb)^4}-0}{t} = \lim_{t\to 0}\frac{t^3ab^2}{t^3(a^2+t^2b^4)} = \lim_{t\to 0}\frac{ab^2}{a^2+t^2b^4} = \frac{ab^2}{a^2} = \frac{b^2}{a}.$$

so $D_uh(0,0)$ exists and that

$$D_uh(0,0) = \frac{b^2}{a} \qquad \text{if } a \neq 0.$$

Consider the sequence $u_n = (\frac{1}{n}, \frac{1}{n})$ and $v_n = (\frac{1}{n^2}, \frac{1}{n})$. We have $u_n \to (0,0)$ as $n \to +\infty$. However

$$k(\frac{1}{n},\frac{1}{n}) = \frac{\frac{1}{n}(\frac{1}{n})^2}{(\frac{1}{n})^2+(\frac{1}{n})^4} = \frac{n}{n^2+1}$$

and

$$k(\frac{1}{n^2},\frac{1}{n}) = \frac{\frac{1}{n^2}(\frac{1}{n})^2}{(\frac{1}{n^2})^2 + (\frac{1}{n})^4} = \frac{1}{2}$$

and $k(u_n) \to 0$ and $k(v_n) \to \frac{1}{2}$ as $n \to +\infty$. Thus

$$\lim_{(x_n,y_n)\to(0,0)} k(x_n,y_n)$$

does not exist. Therefore k is not continuous at $(0,0)$ and hence not differentiable at $(0,0)$.

Exercise (39.E.). Let $f\colon \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$f(x,y) = \begin{cases} 0 & \text{for } (x,y) = (0,0), \\ \frac{xy^2}{x^2+y^2} & \text{for } (x,y) \neq (0,0). \end{cases}$$

Show that the partial derivative of f at $(0,0)$ with respect to any vector $u = (a,b)$ exists and that

$$D_u f(0,0) = \frac{ab^2}{a^2+b^2} \qquad \text{if } (a,b) \neq (0,0).$$

Show that f is continuous but not differentiable at $(0,0)$.

Solution. Since

$$\begin{aligned} \lim_{t\to 0} \frac{f(ta,tb) - f(0,0)}{t} &= \lim_{t\to 0} \frac{\frac{(ta)(tb)^2}{(ta)^2+(tb)^2} - 0}{t} \\ &= \lim_{t\to 0} \frac{t^3ab^2}{t^3(a^2+b^2)} \\ &= \lim_{t\to 0} \frac{ab^2}{a^2+b^2} \\ &= \frac{ab^2}{a^2+b^2}. \end{aligned}$$

so $D_u f(0,0)$ exists and that

$$D_u f(0,0) = \frac{ab^2}{a^2+b^2} \qquad\qquad \text{if } (a,b) \neq (0,0).$$

Since $x^2 + y^2 \geq 2\,|xy|$, so

$$0 \leq \left| \frac{xy^2}{x^2+y^2} \right| \leq \frac{\left|xy^2\right|}{2\,|xy|} = \frac{1}{2}\,|y|\,.$$

Thus

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = 0.$$

Since

$$D_1 f(0,0) = \lim_{\Delta x\to 0} \frac{f(\Delta x,0) - f(0,0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\frac{\Delta x(0)^2}{(\Delta x)^2+0^2} - 0}{\Delta x} = 0$$

and

$$D_2 h(0,0) = \lim_{\Delta y\to 0} \frac{f(0,\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y\to 0} \frac{\frac{(0)(\Delta y)^2}{0^2+(\Delta y)^2} - 0}{\Delta y} = 0$$

so the partial derivatives exist. If f were differentiable at $(0,0)$ then we would have

$$\begin{aligned} D_u f(0,0) &= Df(0,0)(u) \\ &= u_1 D_1 f(0,0) + u_2 D_2 f(0,0) \\ &= u_1(0) + u_2(0) = 0. \end{aligned}$$

However,

$$D_u f(0,0) = \frac{ab^2}{a^2+b^2} \qquad\qquad \text{if } (a,b) \neq (0,0).$$

Therefore f is not diffferntiable at $(0,0)$.

EXERCISE (39.F.). Let $F\colon \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$F(x,y) = \begin{cases} x^2+y^2 & \text{if both } x,y \text{ are rational,} \\ 0 & \text{otherwise.} \end{cases}$$

Show that F is continuous only at the point $(0,0)$ and that it is differentiable here.

SOLUTION. Solution. For any $\epsilon > 0$, there exists $\delta = \sqrt{\epsilon}$ such that if $\|(x,y) - (0,0)\| = \|(x,y)\| = \sqrt{x^2+y^2} < \delta$, then $|F(x,y) - F(0,0)| = \left|x^2+y^2\right| = x^2+y^2 < \delta^2 = \epsilon$. Thus F is continuous at $(0,0)$.

Since

$$D_1F(x,y) = 2x$$

and

$$D_2F(x,y) = 2y$$

are continuous at $(0,0)$, so F is differntiable at $(0,0)$. (Theorem 39.9)

EXERCISE (39.G.). Let $G\colon \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$G(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{x^2+y^2} & \text{for } (x,y) \neq (0,0), \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$$

Show that F is differentable at every point of $\mathbf{R}^2$ but the partial derivative D_1G, D_2G are not bounded (and hence not continuous) on a neighborhood of $(0,0)$.

SOLUTION. Since $\|(xy) - (0,0)\| = \|(x,y)\| = \sqrt{x^2+y^2} < \delta = \epsilon$

$$|G(x,y) - G(0,0)| = \left|(x^2+y^2)\sin\frac{1}{x^2+y^2}\right| \leq \left|x^2+y^2\right| = \sqrt{x^2+y^2}\sqrt{x^2+y^2} < \epsilon\,\|(x,y)-(0,0)\|$$

so F is differentable at every point of $\mathbf{R}^2$.

Since

$$D_1G(x,y) = 2x\sin\frac{1}{x^2+y^2} - \frac{2x\cos\frac{1}{x^2+y^2}}{x^2+y^2}$$

and

$$D_2G(x,y) = 2y\sin\frac{1}{x^2+y^2} - \frac{2y\cos\frac{1}{x^2+y^2}}{x^2+y^2}.$$

Let $u_n = (\frac{1}{2n\sqrt{\pi}}, 0)$. Then

$$2(\frac{1}{2n\sqrt{\pi}})\cos(4\pi n^2)4n^2\pi = 4n\sqrt{\pi} \to \infty$$

as $n \to \infty$.

Similarly, let $v_n = (0, \frac{1}{2n\sqrt{\pi}})$. Applying the same argument above.

EXERCISE (39.W.). Let $A \subseteq \mathbf{R}^p$ and $B \subseteq \mathbf{R}^q$ and let $G\colon A \times B \to \mathbf{R}^r$ to be differentiable at a point (a, b) in $A \times B$. We define $g_1\colon A \to \mathbf{R}^r$ and $g_2\colon B \to \mathbf{R}^r$ to be the "partial maps" at (a, b) given by

$$\begin{aligned} g_1(x) &= G(x, b), \\ g_2(y) &= G(a, y) \end{aligned}$$

for all $x \in A$, $y \in B$. Show that g_1 and g_2 are differentiable at a and b, respectively, and that

$$\begin{aligned} Dg_1(a)(u) &= DG(a, b(u, 0), \\ Dg_2(b)(v) &= DG(a, b)(0, v), \end{aligned}$$

for all $u \in \mathbf{R}^p$, $v \in \mathbf{R}^q$. Moreover, we have

$$DG(a, b)(u, v) = Dg_1(a)(u) + Dg_2(b)(v).$$

[Sometimes $Dg_2(a) \in \mathscr{L}(\mathbf{R}^p, \mathbf{R}^r)$ and $Dg_2(b) \in \mathscr{L}(\mathbf{R}^q, \mathbf{R}^r)$] are called the "block partial derivatives" of G at (a, b) and are denoted by $D_{(1)}G(a, b)$ and $D_{(2)}G(a, b)$.]

Bibliography

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